Modeling the World

By Luis M. Rocha and Santiago Schnell

"*When you can measure what you are speaking of and express it in numbers you know that on which you are discoursing. But if you cannot measure it and express it in numbers. your knowledge is of a very meagre and unsatisfactory kind.*". (Lord Kelvin)

Formalizing Knowledge

As we have seen, the central structure of information is a relation among signs, objects or things, and agents capable of understanding (or decoding) the signs. An AGENT is *informed* by a SIGN about some THING. When we build symbolic abstractions of the World we gain the ability to manipulate and create symbols for aspects of reality we have never actually seen (see figure 1).

Figure1: Big Foot and Unicorn Crossing Road Signs

Once we create symbols, we can also hypothesize relationships among the symbols which we can later check for consistency with what we really observe in the World. By creating relationships among the symbols of things we observe in the World, we are in effect formalizing our knowledge of the World. By formalizing we mean the creation of *rules*, such as verbal arguments and mathematical equations, which define how our symbols relate to one another.

Physics was the first science to construct precise formal rules of the things in the world. Aristotle (484-322 BC) was the first to relate symbols more explicitly to the external world and to successively clarify the nature of the symbol-world (symbol-matter) relation. "In his Physics he proposed that the two main factors which determine an object's speed are its weight and the density of the medium through which it travels. More importantly, he recognized that there could be mathematical rules which could describe the relation between an object's weight, the medium's density and the consequent rate of fall.” [Cariani, 1989, page 52] The rules he proposed to describe this relations were:

1. *For freely falling or freely rising bodies*: speed is proportional to the density of the medium.
2. *In forced motion*: speed is directly proportional to the force applied and inversely proportional to the mass of the body moved

This was the first time that the relationships between observable quantities were hypothesized and used in calculations. Such a formalization of rules as a hypothesis to be tested is what a model is all about. Knowledge is built upon models such as this that sustain our observations of the World.

“While these quantities were expressed in terms of numbers, they were still generally regarded as inherent properties of the objects themselves. It was not until Galileo took the interrelationships of the signs themselves as the objects of study that we even see the beginnings of what was to be progressive dissociation of the symbols from the objects represented. Galileo's insight was that the symbols themselves and their interrelations could be studied mathematically quite apart from the relations in the objects that they represented. This process of abstraction was further extended by Newton, who saw that symbols arising from observation […] are distinct from those involved in representing the physical laws which govern the subsequent motion”. [Cariani, 1989, page 52]

Hertz’ Modeling Paradigm

“In 1894 Heinrich Hertz published *his Principles of Mechanics* which attempted […] to purge mechanics of metaphysical, mystical, undefined, unmeasured entities such as force and to base the theory explicitly on measurable quantities. Hertz wanted to be as clear, rigorous, and concise as possible, so that implicit, and perhaps unnecessary, concepts could be eliminated from physical theories, [which he thought should be based solely on measurable quantities].” [Cariani, 1989, page 54]. Since the results of measurements are symbols, physical theory should be about building relationships among observationally-derived symbols, that is, it should be about building formal *models*, which Hertz called "images”:

> “The most direct and in a sense the most important, problem which our conscious knowledge of nature should enable us to solve is the anticipation of future events, so that we may arrange our present affairs in accordance with such anticipation. As a basis for the solution of this problem we always make use of our knowledge of events which have already occurred, obtained by chance observation or by prearranged experiment. In endeavoring thus to draw inferences as to the future from the past, we always adopt the following process. We form for ourselves images or symbols of external objects; and the form which we give them is such that the

necessary consequents of the images in thought are always the images of the necessary consequents in nature of the things pictured. In order that this requirement may be satisfied, there must be a certain conformity between nature and our thought. Experience teaches us that the requirement can be satisfied, and hence that such a conformity does in fact exist. When from our accumulated experiences we have succeeded in deducing images of the desired nature, we can then in a short time develop by means of them, as by means of models, the consequences in the external world which only arise in a comparatively long time, or as a result of our own interposition. We are thus enabled to be in advance of the facts, and to decide as to present affairs in accordance with the insight so obtained. The images which we here speak are of our conceptions of things. With the things themselves they are in conformity in *one* important respect, namely, in satisfying the above mentioned requirement. For our purpose it is not necessary that they should be in conformity with the things in any other respect whatever. As a matter of fact, we do not know, nor do we have any means of knowing, whether our conceptions of things are in conformity with them in any other than the *one* fundamental respect. [Hertz, 1894 pp. 1-2]"

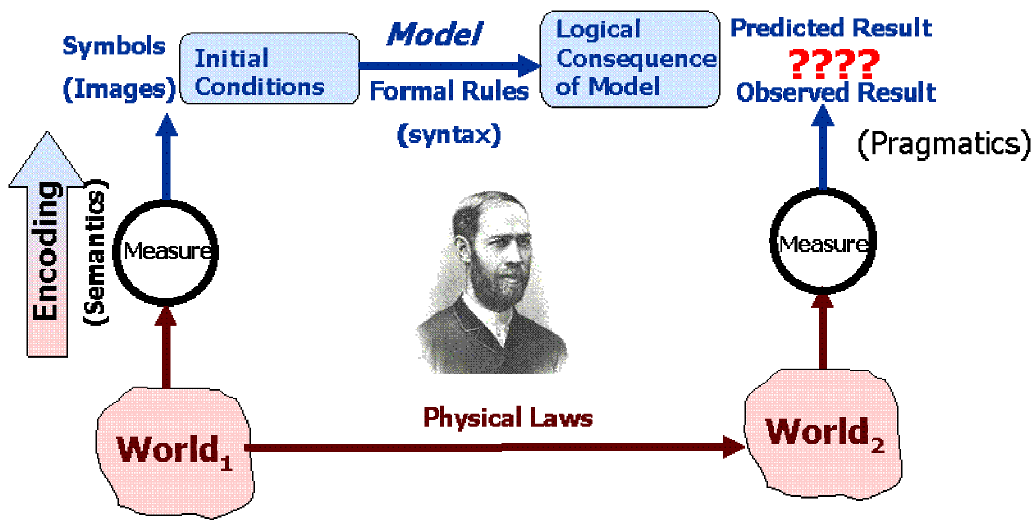

Figure 2: The Modeling Relation between knowledge and reality according to Hertz (adapted from Cariani, 1989)

What is a model?

A model is any complete and consistent set of verbal arguments, mathematical equations or computational rules which is thought to correspond to some other entity, its prototype. The prototype can be a physical, biological, social psychological or other conceptual entity.

The etymological roots of the word model lead us to the Latin word "modulus", which refers to the act of molding, and the Latin word "modus" (a measure) which implies a change of scale in the representation of an entity. The idea of a change of scale, can be interpreted in different ways. As the prototype of a physical, social or natural object, a model represents a change on the scale of abstraction: certain particularities have been removed and simplifications are made to derive a model.

In the natural sciences, models are used as tools for dealing with reality. They are caricatures of the real system specifically build to answer questions about it. By capturing a small number of key elements and leaving out numerous details, models help us to gain a better understanding of reality. Models can be formal and quantitative, as in the strict Hertzian formulation, but they can also be qualitative, if they are not designated to reproduce or predict numerically concrete situations.

According to Albert Einstein "only theory can tell us what to measure and how to interpret it". The meaning is clear: to understand the World in scientific terms it is necessary to build models. Models reduce the apparent complexity of the world into a set of simple rules, and these rules constitute a theory.

The Purpose of Modeling

It is widely believed that the purpose of modeling is to elucidate new concepts (or theories) and describe phenomena through the use of mathematics or computational rules, and to stimulate the creation of new knowledge, as well as the development of new mathematics and computational rules through such studies.

Modeling serves several purposes: Data analysis, interpretation, control, prediction and understanding are central. *Data analysis* and *interpretation* allow hypothesis testing, parameter testing and making statements about the object phrased in a language that only refers to the model. *Prediction*, as Hertz emphasized, is obviously the crucial objective of modeling. Using models, Science attempts to forecast and control consequences at many levels from, for example, drug effects to atmospheric increase in carbon dioxide. *Control* can be viewed as a special kind of prediction answering how to modify or perturb a system to achieve a desired end.

The concept of *understanding* is illusive, but implies a simplification in modeling to the degree that the model is immediately and intuitively understandable. This does not mean that the model used can be understood directly, but that key aspects should be. It could well be possible to have admirable models that are opaque: A complete model of a cell, in all its details, capable of perfectly predicting its behavior would be a major scientific and computational achievement from a prediction standpoint, but it would fail to explain the cell, as it would be in some sense "just" a formal copy of a cell. For a model to increase our understanding of the World, it needs to strip reality from unnecessary details, and capture the fundamental processes that cause the phenomenon we wish to understand with the model. Indeed, presented with the fully detailed model of the cell, a student might rightly ask "How does it work?" Understanding implies some simplification and approximation relative to reality – and its perfect facsimiles.

How are models constructed and judged?

Models (or theories) and experimental observations go hand in hand. Models cannot exist without experiments. The experiment is the basis of the scientific method, while the theory is only relevant once the experimental observations have been made. The role of theory is to reduce complexity. When scientists are faced with the problem of constructing a model, one of the first steps they need to clarify is the level of description in which one is primarily interested with regard to space, time, states and interactions. The appropriate choice of the level of description and scales should be a preliminary decision of causes: the modeler needs to distinguish the phenomena to be considered and those to be ignored. In a model we describe a system in simple, logical terms. Therefore, listing all the essential factors that contribute to the system behavior is the first important step in modeling. Once the model is in place, the scientist needs to test the different scenarios and assumptions, to demonstrate that certain ideas should or cannot be realized, or to give predictions for testing the model. The process of modeling involves carrying out a details system analysis of the model. Because in developing a model we have to build up a small set of assumptions, we can follow up the model rigorously to its conclusions. Then we need to test these conclusions against the empirical facts. After that, we can refine the model's assumptions and go through the loop again, and again. It is important to emphasize that the revision of ideas and the development of models are not necessarily in the direction of greater complexity or increasing the number of parameters. Progress could be towards simplification and reduction of the number of parameters and constants.

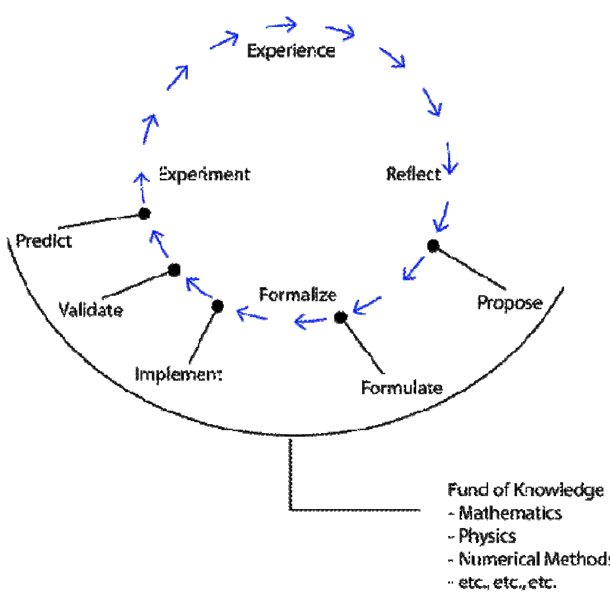

Figure 3: Modeling and the Experimental Process

In summary, the modeling process can be conveniently divided into the following three steps: i) The formulation of the scientific problem in mathematical or computational terms, ii) the solution of the mathematical or computational problem thus created, and iii) the interpretation of the solution and its empirical verification in scientific terms.

Modeling strategies for dummies

Learning how to model systems is knowing what to look for. Modeling problems often require established procedures and knowing what and when to apply them. To identify procedures, you have to be familiar with the problem situation and be able to collect the appropriate information, identify a strategy or strategies and use the strategy appropriately. G. Polya wrote a book titled 'How To Solve It' in 1957. Many of the ideas that worked then continue to work for us now. The steps below are very similar to those expressed in Polya's book. There are four steps involved:

Step 1	**Understand the problem.** We must read it carefully to discover what it asks us to find. This is the unknown. We must discover what facts are given to us. These are the data. Explain the question to other people. Draw a figure. Introduce suitable notation.
Step 2	**Formulate a plan (a model!).** Find the connection between the data and the unknown. You may need to consider auxiliary problems. We develop a suitable plan for modelling our problem, using the data we have been given. We ask ourselves such questions as Is this a familiar problem? Have you seen it before? Do you know a related or analogous problem? Do I have sufficient information to answer it?
Step 3	**Carry out our plan.** Calculate the model using all data and conditions. Do all the calculations, and check them as they go along. Ask: “Can I see it is right?” and then, “Can I prove it is right?”
Step 4	**Looking back.** Examine the solution obtained. Can you check the result? Can you derive the solution differently? Were the modeling predictions correct?

A practical example: Electoral mathematics

Days after the announcement of the electoral results, eager members of the opposition always come with the same question: “How did that guy win?”

The theories abounded: Mr/Mrs X had won the young male vote. Mr/Mrs X has won because voters need some way to keep themselves amused. Mr/Mrs X has won because this country is full of ignorant people and fundamentalists! According to Donald Saari of the University of California, Irvine, there is one theory all the political commentators missed or eager opponents: Mr/Mrs won because of the way we count our votes.

Electoral modelers do not generally study actual elections to analyze the strengths and weaknesses of various voting methods. Modelers usually assume that individual voters are rational. In other words, if they prefer A to B and B to C, they also prefer A to C. If so, then every voter can, in theory, put together a ranking of his or her preferences (possibly with ties). The collection of all such rankings is the electorate's "voter profile". Let us consider an example which mathematicians, such as Donald Saari, like to use in their lectures:

Fifteen people were deciding what beverage to serve at a party: soda, water, or milk. They have collected a voter profile for each member of the group. After counting, they find that six preferred milk first, soda second, and water third. Five preferred water first, soda second, and milk third; four preferred soda first, water second, and milk third. At first, the group tried to decide on a beverage by a simple show of hands. In other words, they employ a plurality vote. Naturally, milk was the winner. But immediately, a member of the group noticed furiously that milk was the last choice of 60% of the voters!

It seems that the plurality vote is not the idea counting method to arrive at a fairer result. Then, some of the party-goers suggested a runoff election. The two top vote-getters in the show of hands, milk and water, were pitted against each other. All the soda lovers switched their votes to water, and so water won, 9-6. Everyone was happy until someone pointed our, "Why are we getting water, when 10 of us would rather have soda?" The party never took place, because the organizers continue arguing over the beverage to serve and could reach a decision.

From this hypothetical election, students always conclude that the outcome of an election depends from the counting procedure, but not the wishes of the voters. Unfortunately, what most of the students doe not seems to understand is that there is more than one possible voting system, and they have competing virtues. Of course, if you take into account the competing virtues of the voting methods, you can use the right method for the problem you would like to solve; for example, what beverage is the winner based on the construction of a voter profile?

Recently, Donald Saari demonstrated mathematically that a voting method known as the Borda count is the only electoral method that treats all voter profiles as ties, and therefore it would be the fairest to employ in our beverage example. In the Borda count, each voter ranks all the candidates. If there are n candidates, the top one on the list receives *n* points, the second receives (*n* - 1) points, and so on. For the tally, the points from all the voters are added up. Then add the total votes for each candidate. The Borda count is commonly used in sports polls in the form of run a round-robin tournament, with each candidate going up against all others in one-on-one races. Under the Borda count, milk would be the winner in our example.

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