Chaos and the Logistic Map
until now

- **Class Book**

- **Lecture notes**
  - Chapter 1: What is Life?
  - Chapter 2: The logical Mechanisms of Life
  - Chapter 3: Formalizing and Modeling the World
    - posted online @ http://informatics.indiana.edu/rocha/i-bic

- **Papers and other materials**
  - Logical mechanisms of life (optional for ISE 483)
  - **Optional**
    - Prusinkiewicz and Lindenmeyer [1996] *The algorithmic beauty of plants*.
      - Chapter 1
      - Chapters 10, 11, 14 – Dynamics, Attractors and chaos
Lab Assignments: 35% (ISE-483), 25% (SSIE-583)
- Complete 5 (best 4 graded) assignments based on algorithms presented in class
  - Lab 2: February 20\(^{th}\) : *L*-Systems (Assignment 2)
    - Due February 27\(^{th}\)
  - Lab 3: Cellular Automata and Boolean Networks (Assignment 3)
    - March 13\(^{th}\)

SSIE – 583 -Presentation and Discussion: 25%
- Present and lead the discussion of an article related to the class materials
  - Enginet students post/send video or join by Zoom
- Next Presentation March 13\(^{th}\) or 20\(^{th}\)
    - Presented by David Birk
The logistic map

- **Demographic model**
  - introduced by Pierre François Verhulst in 1838

- **Continuous state-determined system**
  - Memory of the previous state only

- **Observations**
  - $X=0$: population extinct
  - $X=1$: Overpopulation, leads to extinction

The logistic map is a quadratic equation given by:

$$x_{t+1} = rx_t(1 - x_t)$$

where:

- $x \in [0,1]$ represents the population size,
- $r \in [0,4]$ is the reproduction rate.
The logistic map

- **Demographic model**
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  - X=1: Overpopulation, leads to extinction

\[ x_{t+1} = rx_t(1 - x_t) \]

\[ x \in [0, 1] \]
\[ r \in [0, 4] \]
The logistic map is defined by the equation:

$$x_{t+1} = rx_t (1 - x_t)$$

where:

- $x \in [0,1]$
- $r \in [0,4]$

The derivative of the logistic map is:

$$f'(x) = r(1 - 2x)$$

- If $|f'(x)| < 1$ then the fixed points are stable.
- If $|f'(x)| > 1$ then the fixed points are unstable.

Fixed-point attractors are found by solving:

$$f(x) = x \iff rx(1-x) = x \iff x(r(x-1)+1) = 0$$

The fixed points are:

- $x = 0$ or
- $$x = 1 - \frac{1}{r}$$
logistic map

$r \leq 1$ (population goes extinct)

$x = 0 \lor x = 1 - \frac{1}{r}$

Stable attractor = population extinction

$f'(x) = r(1 - 2x)$,

\[
\begin{cases} 
|f'(x)| < 1 \Rightarrow \text{stable} \\
|f'(x)| > 1 \Rightarrow \text{unstable}
\end{cases}
\]

$x = 0 \Rightarrow |f'(x)| = |r(1 - 2x)| = r,$

\[
\begin{cases} 
 r < 1 \Rightarrow \text{stable} \\
 r > 1 \Rightarrow \text{unstable}
\end{cases}
\]
$1 \leq r \leq 3$

Stable attractor = $1 - 1/r$

$x = 0 \Rightarrow |f'(x)| = r$

$x = 1 - \frac{1}{r} \Rightarrow |f'(x)| = |2 - r|$

$f'(x) = r(1 - 2x), \begin{cases} |f'(x)| < 1 \Rightarrow \text{stable} \\ |f'(x)| > 1 \Rightarrow \text{unstable} \end{cases}$
logistic map

\[ x = 0 \lor x = 1 - \frac{1}{r} \]

\[ f'(x) = r(1 - 2x), \begin{cases} |f''(x)| < 1 \Rightarrow \text{stable} \\ |f''(x)| > 1 \Rightarrow \text{unstable} \end{cases} \]

\[ x = 1 - \frac{1}{r} \Rightarrow |f''(x)| = 2 - r \]

\[ x = 0 \Rightarrow |f''(x)| = r \]

Limit cycle
\[ f(f(x)) = x \]

Stable attractor = 2-point limit-cycle (oscillation)
logistic map

$3 \leq r \leq 4$ ($3.44 \leq r \leq 3.54$)

$x = 0 \lor x = 1 - \frac{1}{r}$

$f(f(f(f(x)))) = x$

Limit cycle

Stable attractor = 4-point limit-cycle
The logistic map is defined by the equation:

$$x_{n+1} = r x_n (1 - x_n)$$

where $r$ is the growth rate parameter.

For $r = 4$, the logistic map exhibits chaotic behavior, which is characterized by:
- Deterministic:
  - The system is completely determined by its initial conditions and parameters.
- Chaotic:
  - The system is highly sensitive to initial conditions, leading to unpredictable long-term behavior.
- Sensitive:
  - Small changes in initial conditions can lead to vastly different outcomes.
- Ergodic:
  - The system's long-term behavior can be studied by averaging over different initial conditions.
logistic map

movie
bifurcation map

Period doubling

Bifurcation Diagram of Logistic Map

\[ 1 - \frac{1}{r} \]
logistic map

bifurcation map

Chaotic

Deterministic (not random)

Sensitive

ergodic
bifurcation map: cycle of 3

logistic map

\[ f(f(f(x))) = x \]

\[ 1 - \frac{1}{r} \]

Period doubling
readings

- **Class Book**
  
    - Chapter 2.

- **Lecture notes**
  
  - Chapter 1: What is Life?
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- **Papers and other materials**
  
  - Optional
      - Chapter 2, all sections
      - Chapter 7, sections 7.3 – Cellular Automata
      - Chapter 8, sections 8.1, 8.2, 8.3.10
      - Chapters 10, 11, 14 – Dynamics, Attractors and chaos