lecture 6


Dynamical Systems,
Attractors, and
Deterministic Chaos
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key events coming up

- Labs: 35% (ISE-483)
  - Complete 5 (best 4 graded) assignments based on algorithms presented in class
    - Lab 2 : February 24th
      - L-Systems (Assignment 2)
        - Delivered by Rik Pardun
        - **Due: March 3rd**
    - Lab 3: March 11th
      - Cellular Automata and Boolean Networks (Assignment 3)
        - Delivered by Kaeli Ahn and Erik Fiolkoski
        - Due: March 24th
- SSIE – 583 -Presentation and Discussion: 25%
  - Present and lead the discussion of an article related to the class materials
    - Enginet students post/send video or join by Zoom

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schedule

- March 3rd, 2025
  - Emad Abed
    - Scheffer, Marten, et al. "Early-warning signals for critical transitions." *Nature* **461**.7260 (2009): 53.
      - Optional: Leemput, Ingrid A van de, Marieke Wichers, Angélique O J Cramer, Denny Borsboom, Francis Tuerlinckx, Peter Kuppens, Egbert H van Nes, et al. "Critical Slowing down as Early Warning for the Onset and Termination of Depression." *Proceedings of the National Academy of Sciences of the United States of America* **111**, no. 1 (January 2014): 87–92.
  - Shayan Esfarayeni
    - Manicka, S., M. Marques-Pita, and L.M. Rocha [2022]. Effective connectivity determines the critical dynamics of biochemical networks. *Journal of the Royal Society Interface*. **19**(186):20210659.
  - Rik Pardun
    - Lindgren, K. [1991]."Evolutionary Phenomena in Simple Dynamics." In: *Artificial Life II*. Langton et al (Eds). Addison-wesley, pp. 295-312.
      - Optional: Hinton, G.E. and S.J. Nowlan [1987]."How learning can guide evolution." *Complex Systems*. **1**, pp.495-502.
  - Kaeli Ahn
    - Papadopoulou, Marina, Hanno Hildenbrandt, Daniel W. E. Sankey, Steven J. Portugal, and Charlotte K. Hemelrijk. "Self-Organization of Collective Escape in Pigeon Flocks." Edited by Feng Fu. *PLOS Computational Biology* **18**, no. 1 (January 10, 2022): e1009772.
- April 22, 2025
  - Rik Pardun
    - Conrad, M. [1990]. "The geometry of evolution." *Biosystems* **24**: 61-81.
  - Ngo Tuan Kiet
    - Garg, Shivam, Kirankumar Shiragur, Deborah M. Gordon, and Moses Charikar. "Distributed Algorithms from Arboreal Ants for the Shortest Path Problem."*Proceedings of the National Academy of Sciences* **120**, no. 6 (February 7, 2023): e2207959120.

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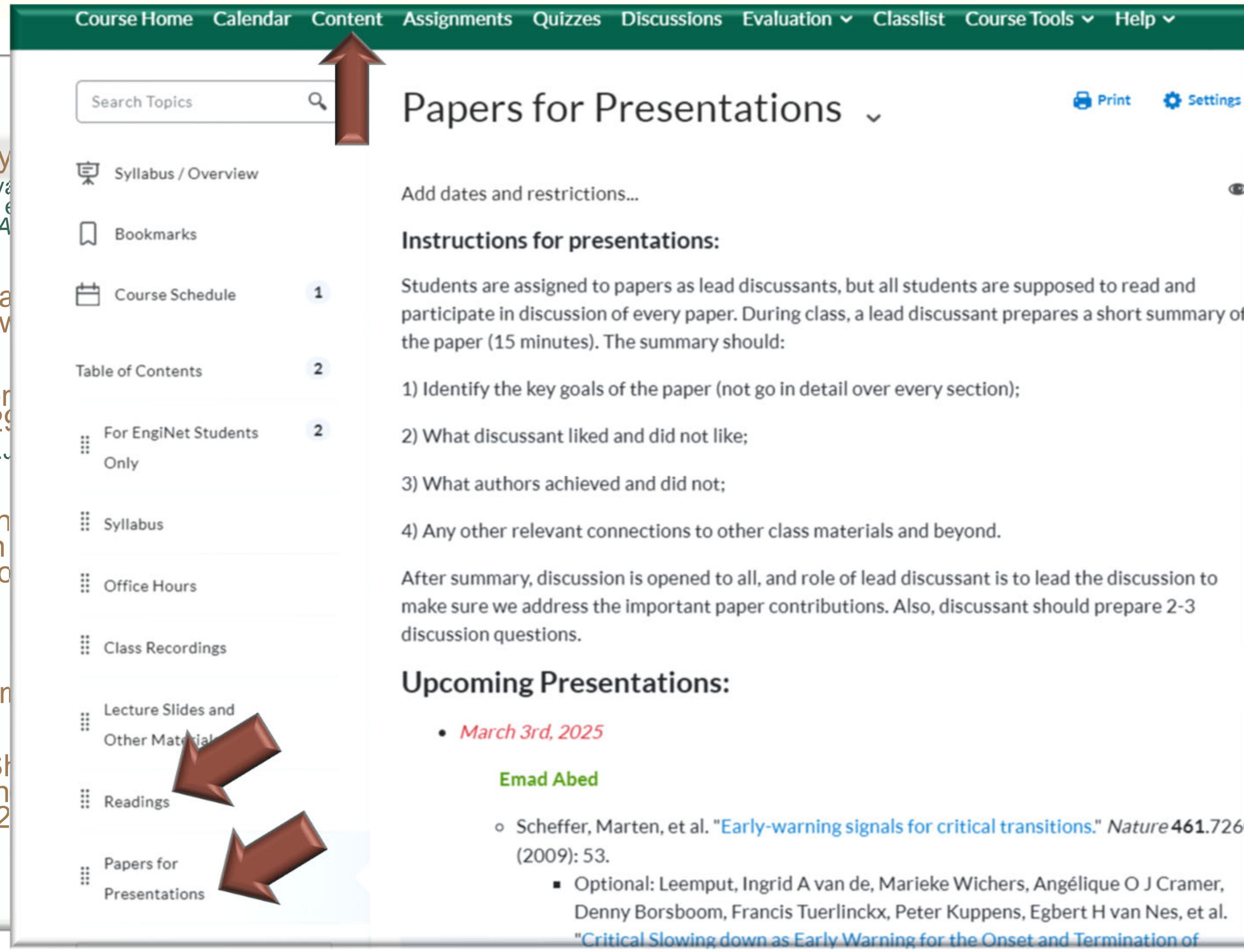

- Projects
  - **Due by May 7th** in Brightspace, “Final Project 483/583” assignment
    - ALIFE 2025
      - Not necessarily to submit to actual conference due date
        - May 4 full paper, July 4, abstract
      - https://2025.alife.org/
      - Max 8 pages, author guidelines:
      - https://2025.alife.org/calls#paper-call
      - MS Word and Latex/Overleaf templates
    - Preliminary ideas **by March 7**
      - Submit to “Project Idea” assignment in Brightspace.
  - Individual or group
    - With very definite tasks assigned per member of group

ALIFE 2025

Tackle a real problem using bio-inspired algorithms, such as those used in the labs.

Reusing and expanding labs is highly encouraged.

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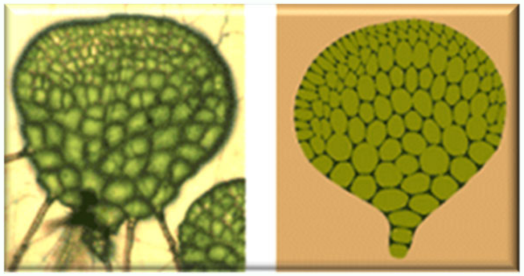
Additional information

- Class Book
  - Floreano, D. and C. Mattiussi [2008]. *Bio-Inspired Artificial Intelligence: Theories, Methods, and Technologies*. MIT Press. Preface, Sections 4.1, 4.2. **Chapter 2.**
    - Nunes de Castro, Leandro [2006]. *Fundamentals of Natural Computing: Basic Concepts, Algorithms, and Applications*. Chapman & Hall. Chapter 1**,** pp. 1-23. Sections **7.1 to 7.4** , Appendix B.3.1. **Chapter 2**, **Sections 8.1, 8.2, 8.3.10**
- Lecture notes
  - Chapter 1: What is Life?
  - Chapter 2: The logical Mechanisms of Life
  - Chapter 3: Formalizing and Modeling the World
    - posted online @ casci.binghamton.edu/academics/i-bic



- Papers and other materials
  - Optional
    - Prusinkiewicz and Lindenmeyer [1996] The algorithmic beauty of plants.
      - Chapter 1
    - Flake's [1998], *The Computational Beauty of Life*. MIT Press.
      - Chapters 1, 5, 6 (7-9).
      - Chapters **10, 11, 14**

modeling similarities across nature

- self-similar structures
  - Trees, plants, clouds, mountains
    - morphogenesis
  - Mechanism
    - Iteration, recursion, feedback
- dynamical systems and unpredictability
  - From limited knowledge or inherent in nature?
  - Mechanism
    - Chaos, measurement
- self-organization, collective behavior, emergence
  - Complex behavior from collectives of many simple units or agents
    - cellular automata, dynamical networks, morphogenesis, swarms, brains, social systems
  - Mechanism
    - Parallelism, multiplicity, multi-solutions, redundancy
- evolution
  - Adaptation, learning, social evolution
  - Mechanism
    - Reproduction, transmission, variation, selection, Turing's tape
- Network causality (heterogenous complexity)
  - Behavior derived from many inseparable sources
    - Immune system, anticipatory systems, brain-body-environment-culture, embodiment, epigenetics, culture
  - Mechanism
    - Modularity, control, hierarchy, connectivity, stigmergy, redundancy

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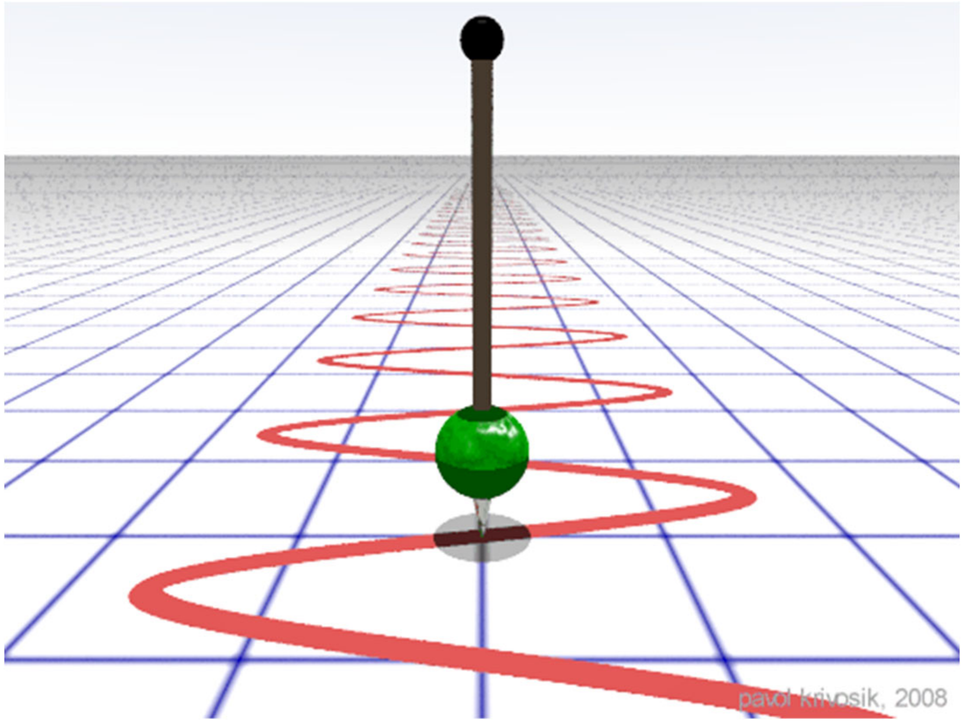
bodies in motion

- Mathematical models of systems containing the rules describing the way some quantity undergoes a change in time
    - What changes in time
        - a variable
            - Position, quantity, concentration
    - How does something change in time
        - Deterministic rules that define change
            - Set of differential equations defining rates of change



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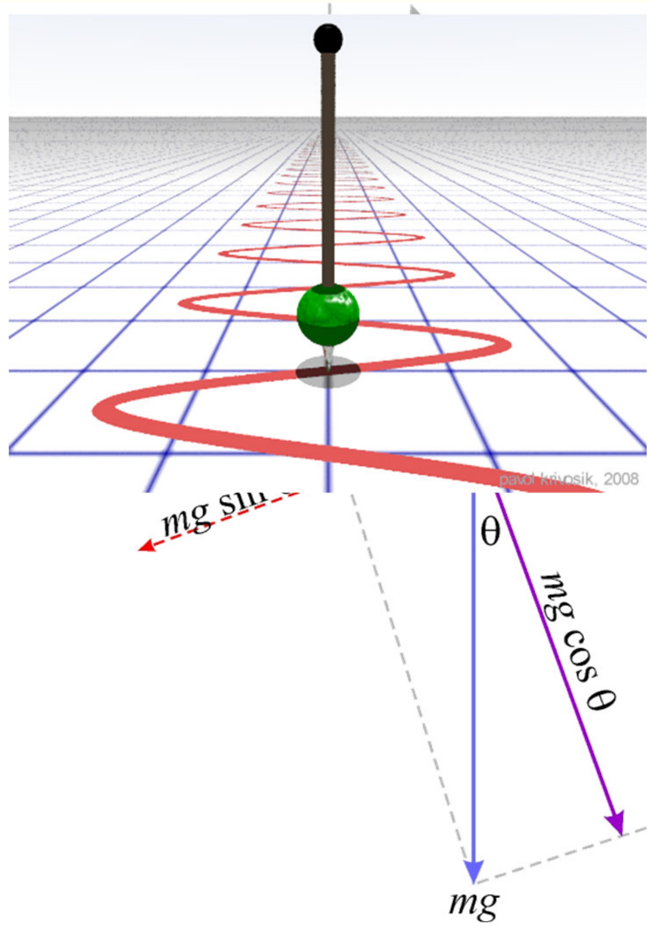
gravitational pendulum example

- What changes in time
  - a variable
    - Angle
  - Rules that define change
    - Set of differential equations defining rates of change



$$F = mg \sin\theta = ma$$

$$a = g \sin\theta$$

$$a = \frac{d^2 s}{dt^2} = l \frac{d^2\theta}{dt^2}$$

$$l \frac{d^2\theta}{dt^2} = g \sin\theta$$

$$\boxed{l \frac{d^2\theta}{dt^2} - g \sin\theta = 0}$$

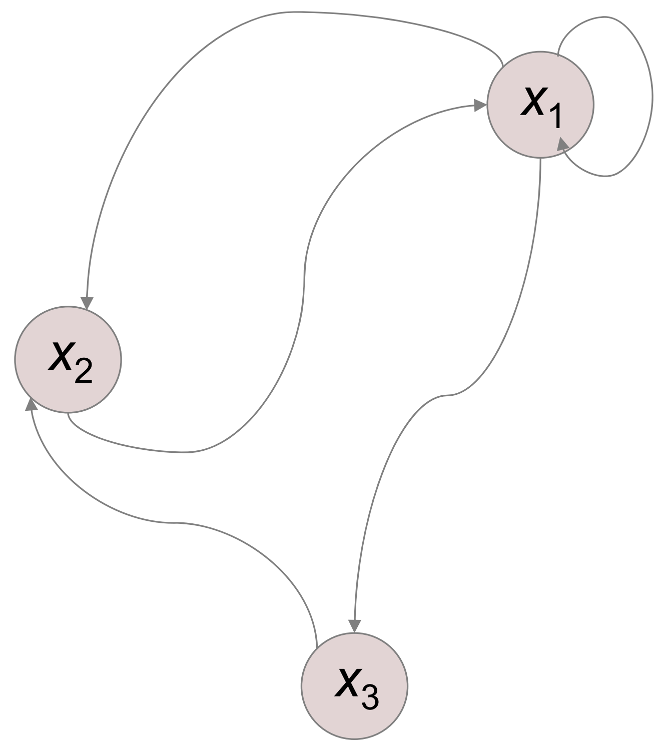
chemical reaction example

- What changes in time
  - a variable
    - concentrations
  - Rules that define change
    - Set of differential equations defining rates of change

$$\frac{dx_1}{dt} = f_1(x_1, x_2) = x_1 - K_1 x_2$$

$$\frac{dx_2}{dt} = f_2(x_1, x_3) = x_1^2 + K_2 x_3$$

$$\frac{dx_3}{dt} = f_3(x_1) = K_3 x_1$$

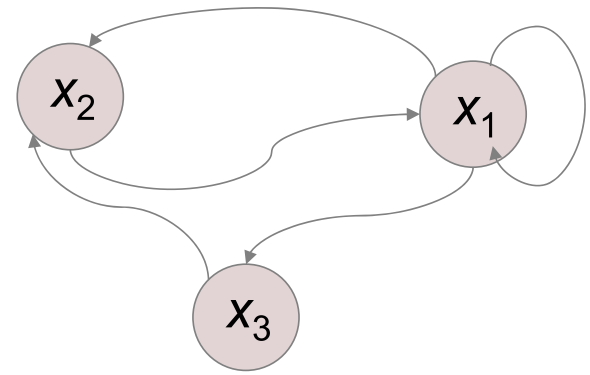


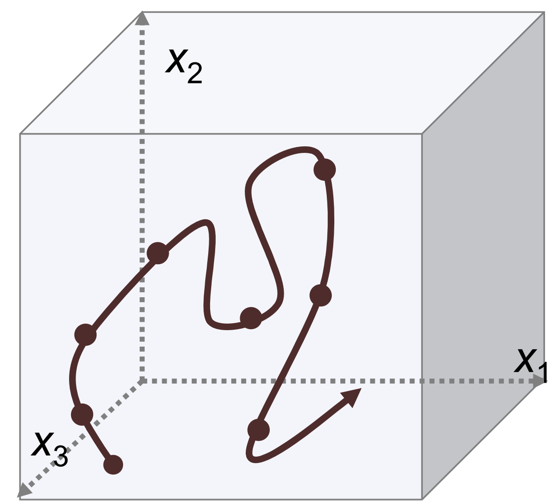
phase or state-space

- Map of variables in time
  - Time is parameter
    - Trajectory (orbit) in state space

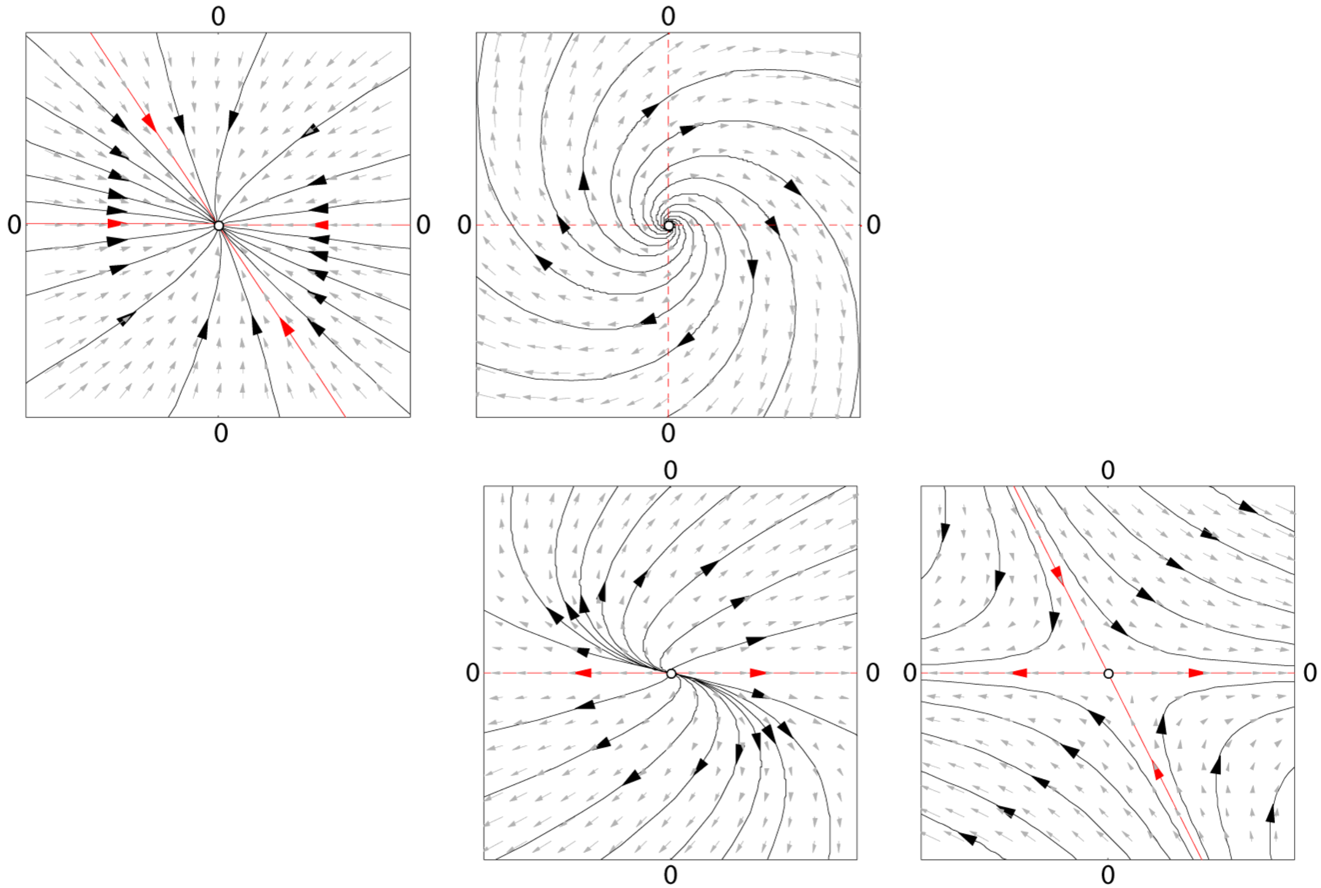
$$X(t) = \left(x_1(t), x_2(t), x_3(t)\right)$$

- Continuous (reversible) systems
  - Only one trajectory passes through each point of a state-space
    - State-determined system
    - 2 points on different trajectories will always be on different trajectories
      - Albeit arbitrarily close
    - Not true in discrete systems
  - Determinism, strict causality
    - Laplace





vector fields represent basins of attraction in phase-space



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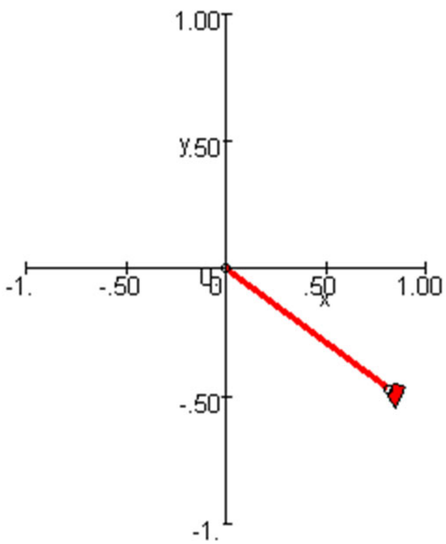
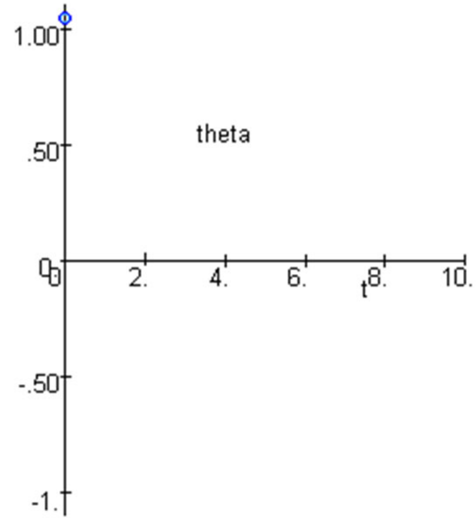
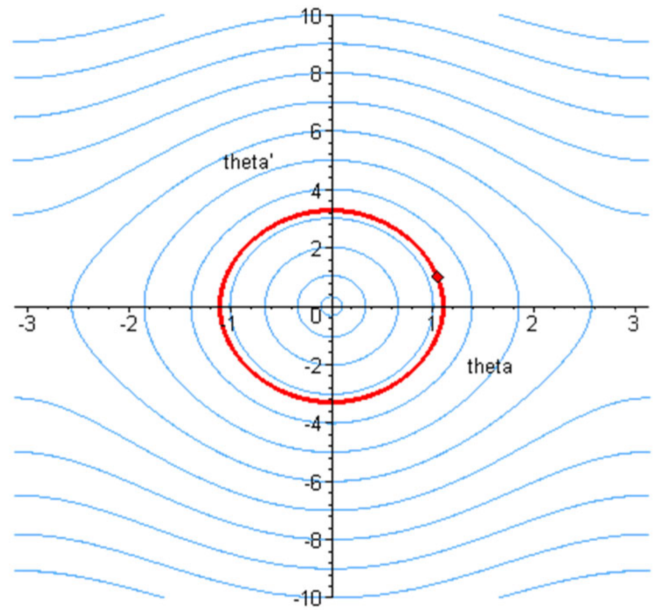
phase space







$$X(t) = \left(\theta(t), \dot{\theta}(t)\right)$$

displacement and velocity

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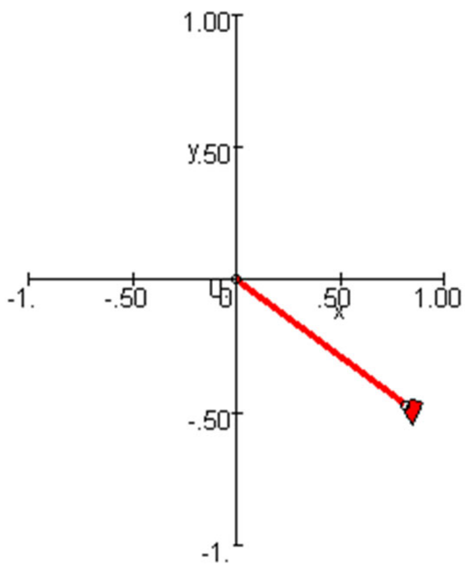
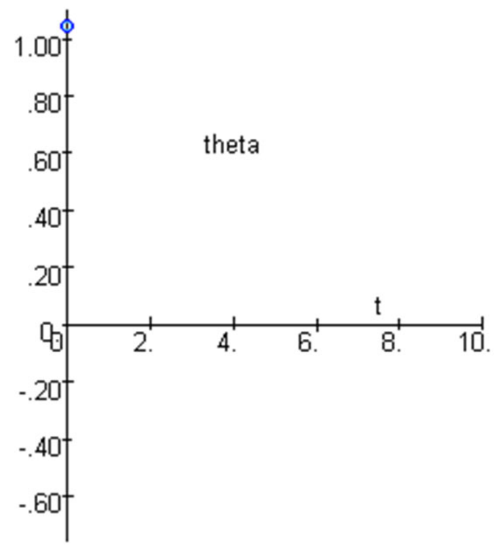
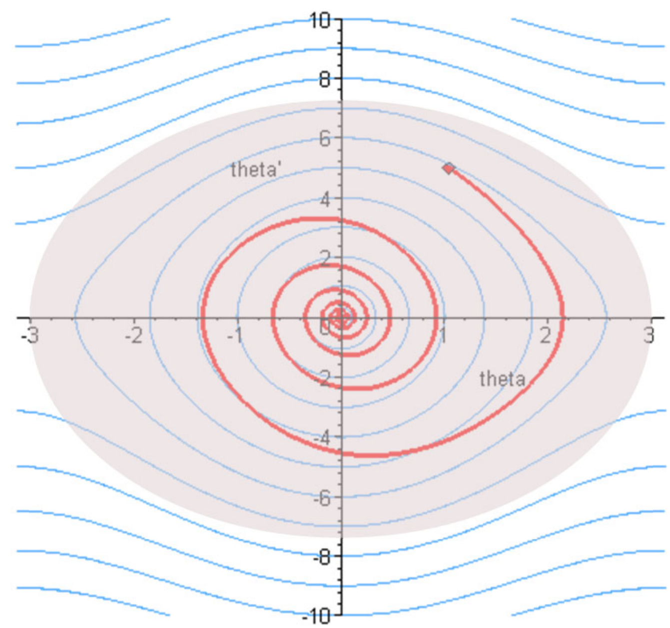
attractor behavior: where motion leads to





Fixed-point behavior

(0-dimensional attractor)



volumes of phase space to which the system converges after a long enough time

Basin of attraction

Volume of the phase-space defined by all trajectories leading into the attractor

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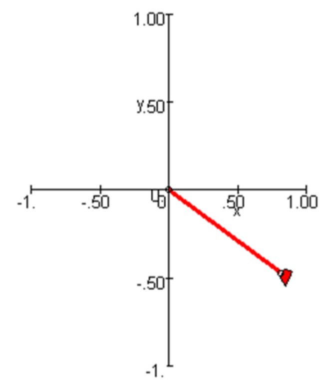
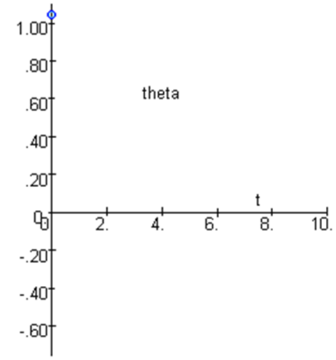
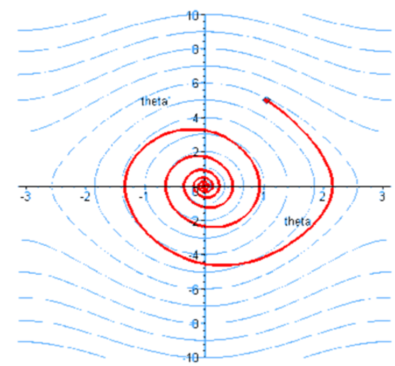
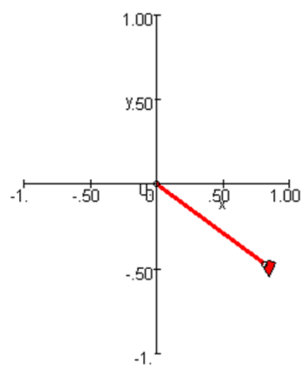
why the attractor behavior?

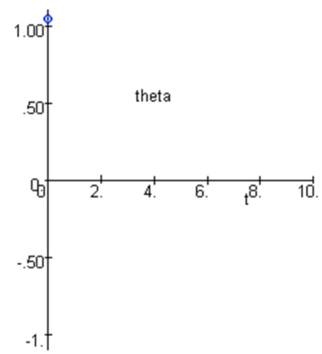




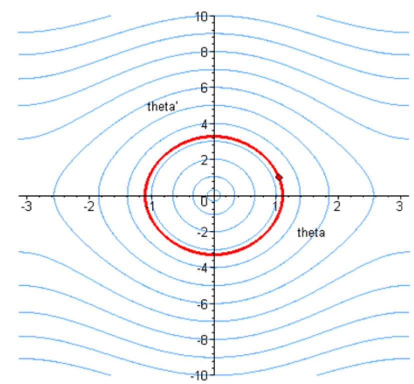
- Energy dissipation (thermodynamic systems)
  - Friction, thermodynamic losses, loss of material, etc.
  - Volume contraction in phase-space
    - System tends to restrict itself to small basins of attraction
  - Self-organization
    - Dissipative systems (Prigogine)





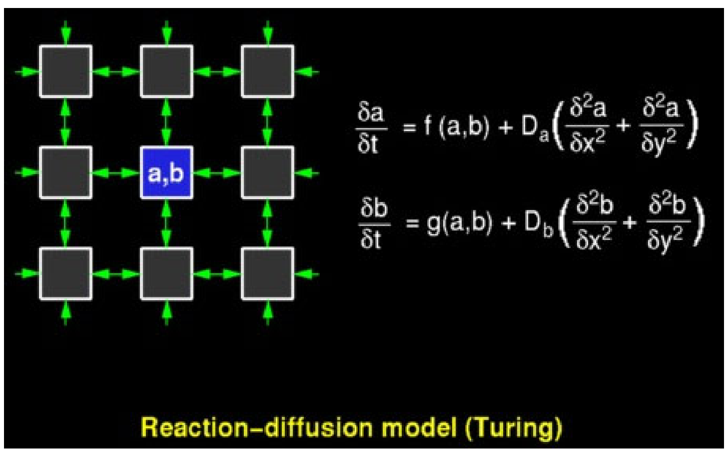


- Hamiltonian systems
  - Frictionless, no attractors
  - Conservation of energy
  - ergodicity



simple dynamics yield complex patterns

- Morphogenesis
  - development of the structure of an organism or part
    - phenotype develops in time under the direction of the genotype + dynamic constraints
  - The process in complex system-environment exchanges that tends to elaborate a system's given form or structure.
- Fischer (1924)
  - Reaction-diffusion equation
    - Propagation of a gene a population
- Nicolas Rashevsky
  - Embryogenesis
- Alan Turing
  - spent the last few years of his life developing his morphogenetic theory and using the new computer to generate solutions to reaction-diffusion systems.



Turing, A. M. [1952] “The chemical basis of morphogenesis”. *Phil. Trans. R. Soc. Lond. B* **237**, 37–72

two homogeneously distributed substances within a certain space, one “locally activated” and the other capable of “long-range inhibition,” can produce novel shapes and gradients.

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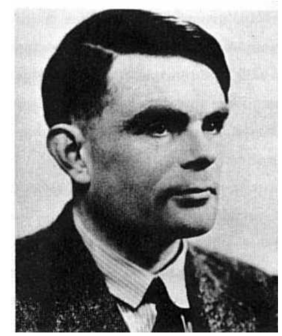
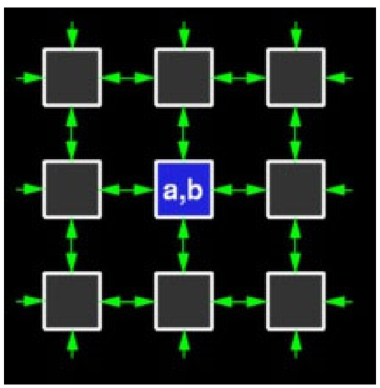
simple dynamics yield complex patterns

- Reaction-diffusion model
  - Stable tension between production and transformation
    - When balance is disturbed, tension restores balance:
  - Metaphor
    - Island populated by cannibals and (celibate) missionaries.
    - Missionaries do not reproduce, but can recruit and die (transform)
    - Cannibals reproduce and die (produce)
    - Two missionaires convert a cannibal leading to tension between production and transformation





By Kele W. Cable

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{Rate of concentration change}} = \overbrace{\underbrace{F(u,v)}_{\text{Production}} - \underbrace{d_u v}_{\text{Degradation}}}^{} + \underbrace{D_u \Delta u}_{\text{Diffusion}}$$

$$\frac{\partial v}{\partial t} = G(u,v) - d_v v + D_v \Delta v$$

Reaction

two homogeneously distributed substances within a certain space, one "locally activated" and the other capable of "long-range inhibition," can produce novel shapes and gradients.

substance interactions depend on just four variables per *morphogen* – the rate of production, the rate of degradation, the rate of diffusion between neighbors, and the strength of activating/inhibiting interactions.

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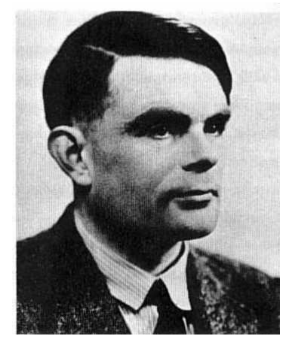

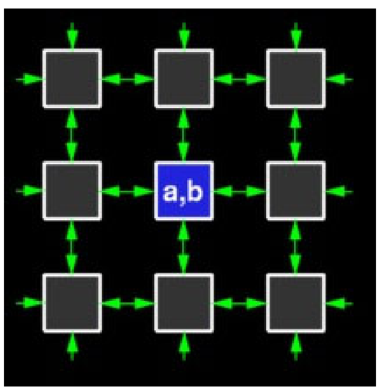

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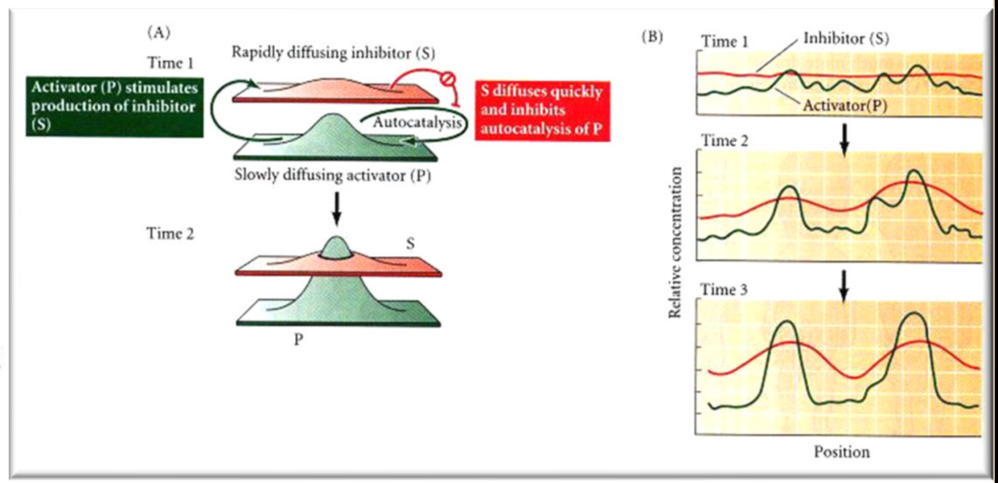

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modeling nature

- Reaction-diffusion model
  - Stable tension between production and transformation
    - When balance is disturbed, tension restores balance:
  - Metaphor
    - Island populated by cannibals and (celibate) missionaries.
    - Missionaries do not reproduce, but can recruit and die (transform)
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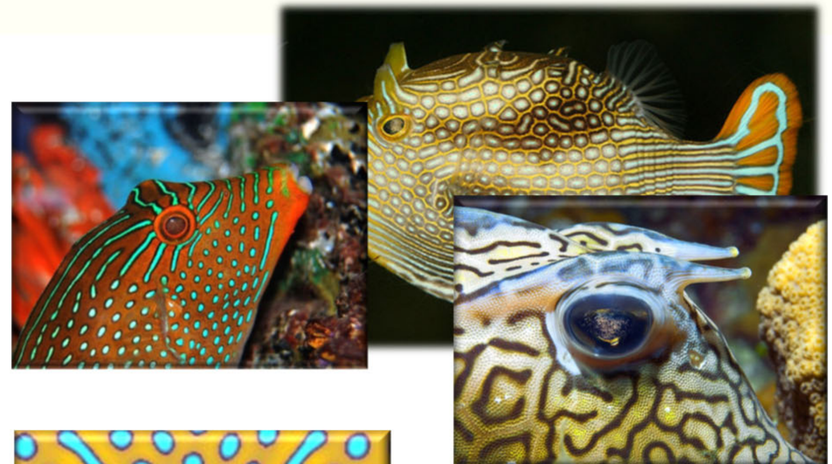

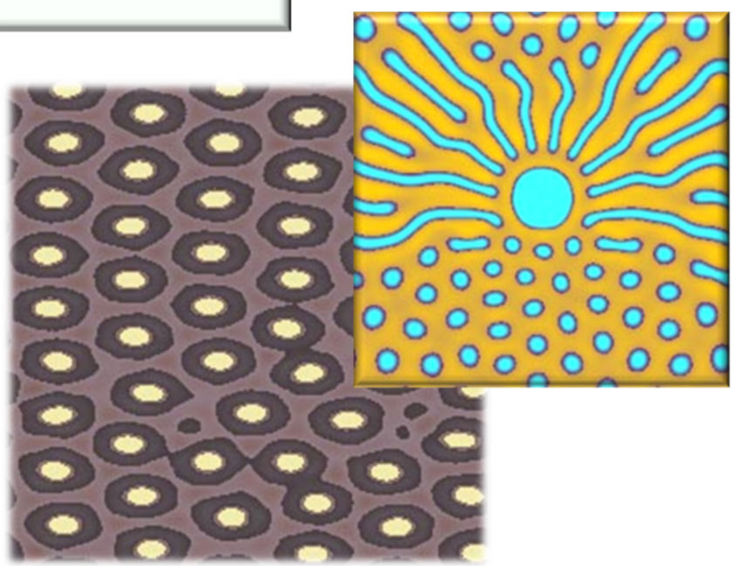

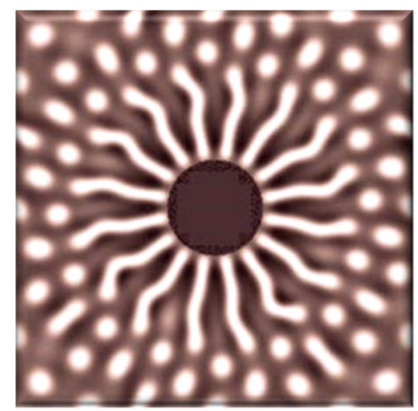

Allen Sanderson

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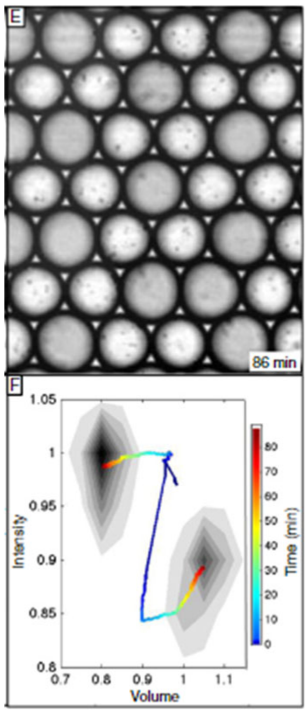
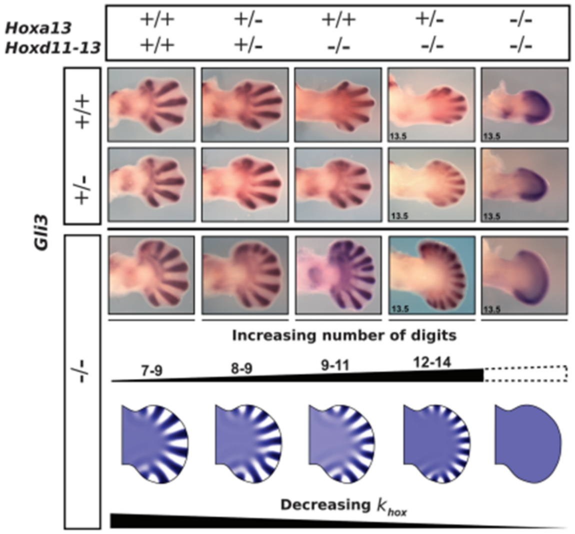
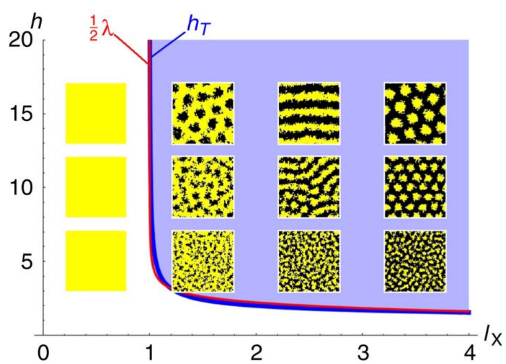
Turing morphogenesis

in biology, chemistry, and complex systems science







Gene expression of digit determination (in mouse)

Sheth et al [2012]. "Hox Genes Regulate Digit Patterning by Controlling the Wavelength of a Turing-Type Mechanism." *Science.* **338** (6113): 1476–80.

Validation of predicted patterns (in abiological droplets)

Tompkins et al [2014]. "Testing Turing's Theory of Morphogenesis in Chemical Cells." *PNAS* **111** **(**12) : 4397–4402.

Revising the model with biological evidence (in zebrafish)

Bullara, D, and Y De Decker [2015]. "Pigment Cell Movement Is Not Required for Generation of Turing Patterns in Zebrafish Skin." *Nature Communications* **6** : 6971.

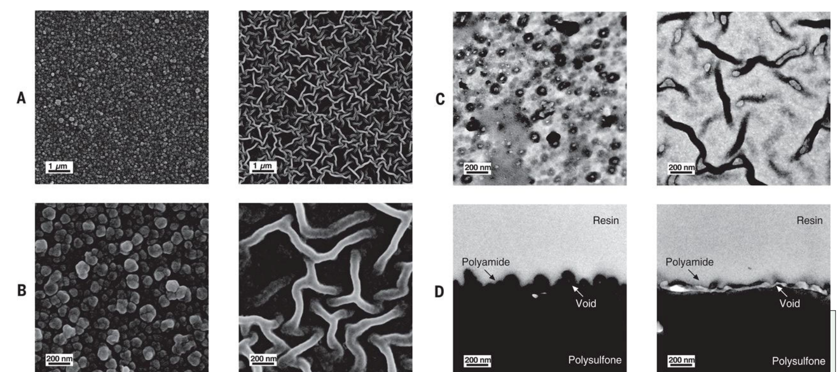
Turing-type polyamide membranes for water purification

Tan, et al [2018]. "Polyamide Membranes with Nanoscale Turing Structures for Water Purification." *Science* **360** **(**6388): 518–21.

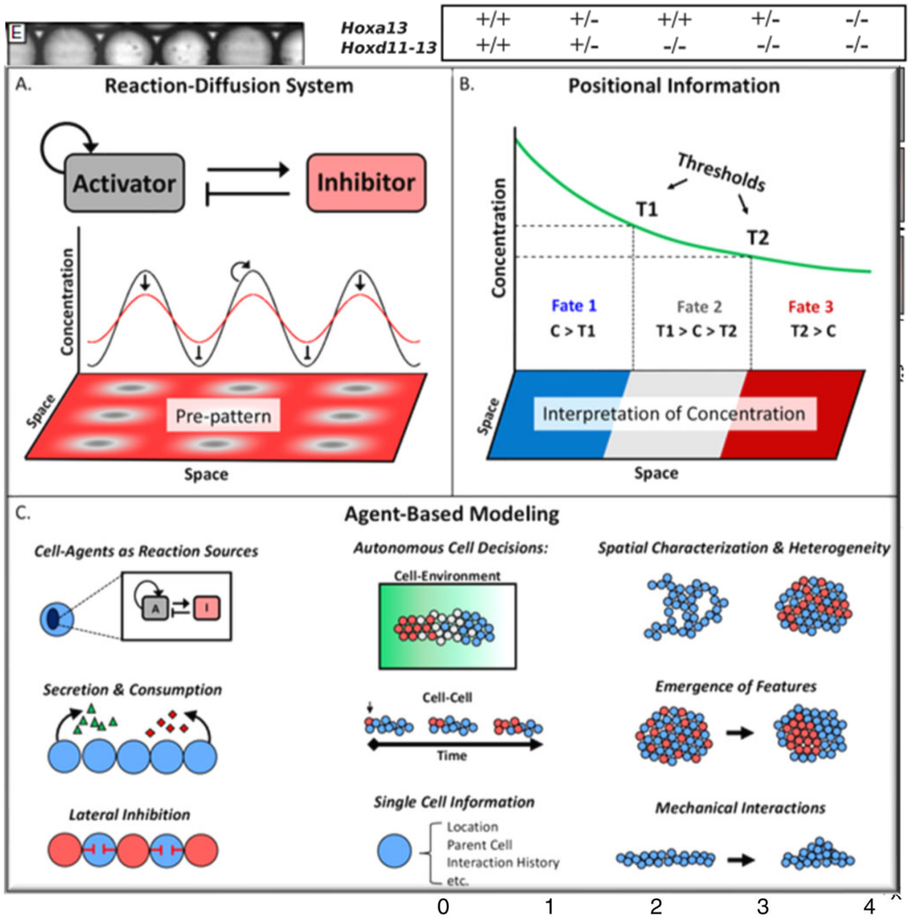
Expanding theoretical models (ABM and others)

Glen et al [2019] "Agent-Based Modeling of Morphogenetic Systems: Advantages and Challenges." *PLOS Computational Biology* **15** (3): e1006577.

Steinbock, Wackerbauer, and Horváth [2019]. "Nonlinear Chemical Dynamics and Its Interdisciplinary Impact: Dedicated to Ken Showalter on the Occasion of His 70th Birthday." *Chaos: An Interdisciplinary Journal of Nonlinear Science* **29** (8): 080401.



in biology, chemistry, and complex systems science



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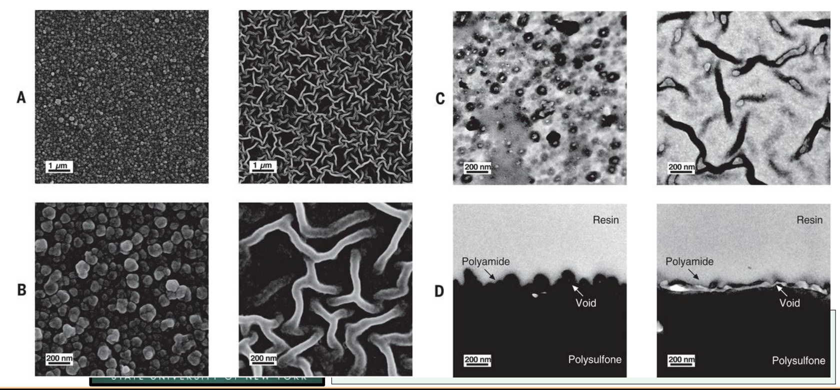
Expanding theoretical models (ABM and others)

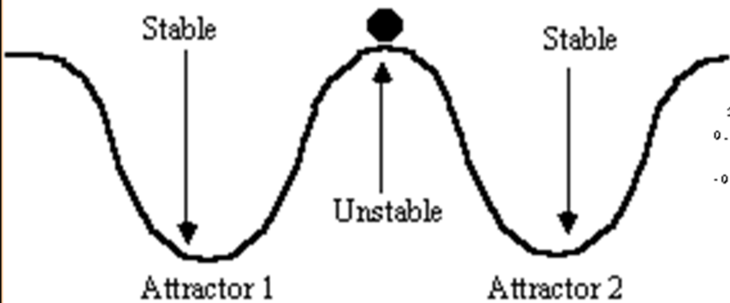
Glen et al [2019] "Agent-Based Modeling of Morphogenetic Systems: Advantages and Challenges." *PLOS Computational Biology* **15** (3): e1006577.

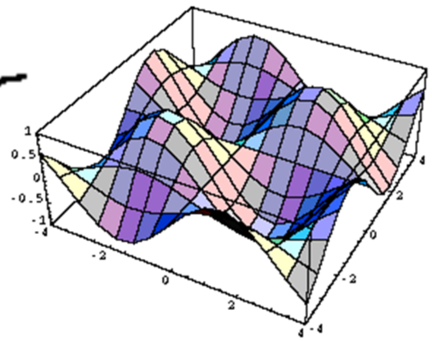
Steinbock, Wackerbauer, and Horváth [2019]. "Nonlinear Chemical Dynamics and Its Interdisciplinary Impact: Dedicated to Ken Showalter on the Occasion of His 70th Birthday." *Chaos: An Interdisciplinary Journal of Nonlinear Science* **29** (8): 080401.

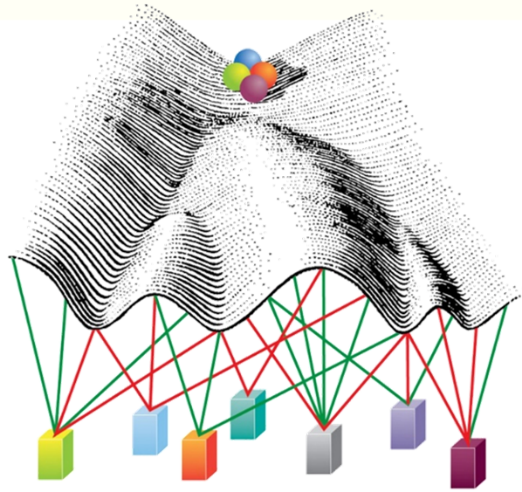


(energy) landscape metaphor

- Phase-space as landscape
  - State of the system as a drop of water released in hills and valleys

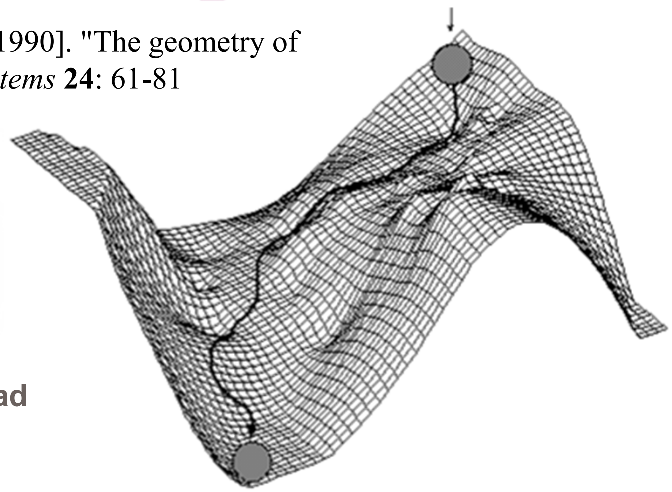




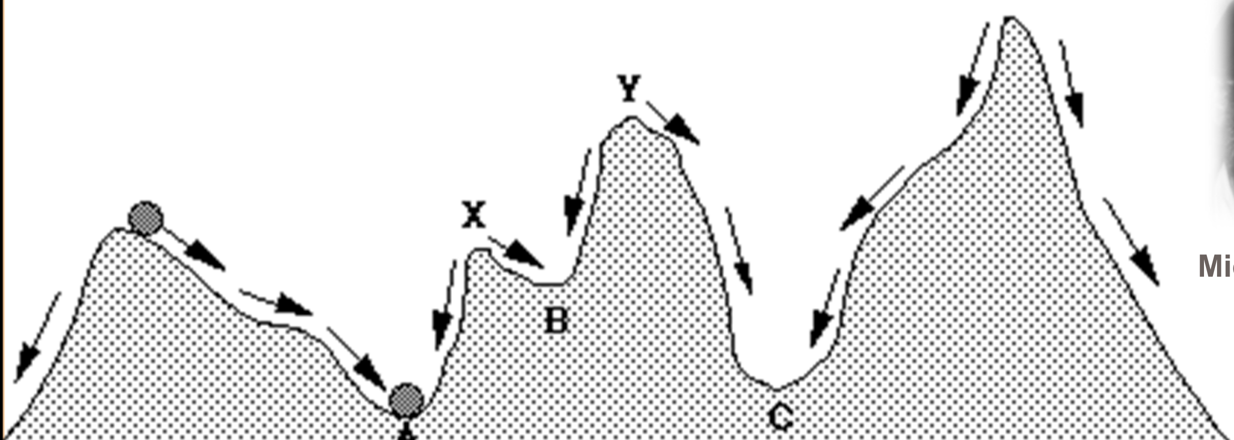


See: Conrad, M. [1990]. "The geometry of evolution.“ *Biosystems* **24**: 61-81

Waddington CH (1942). *Nature*.**150** (3811):563–565



Michael Conrad



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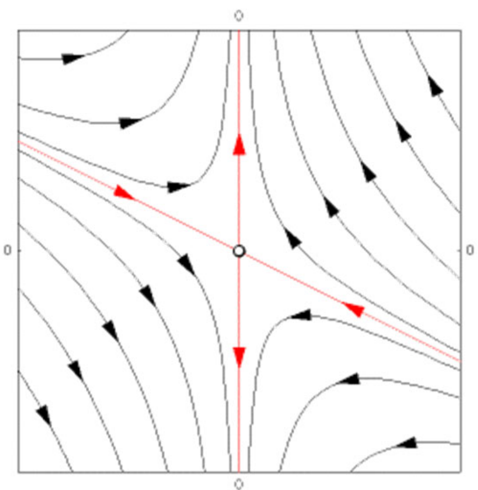
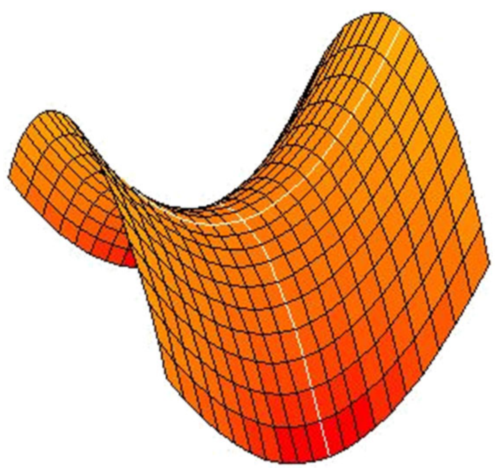
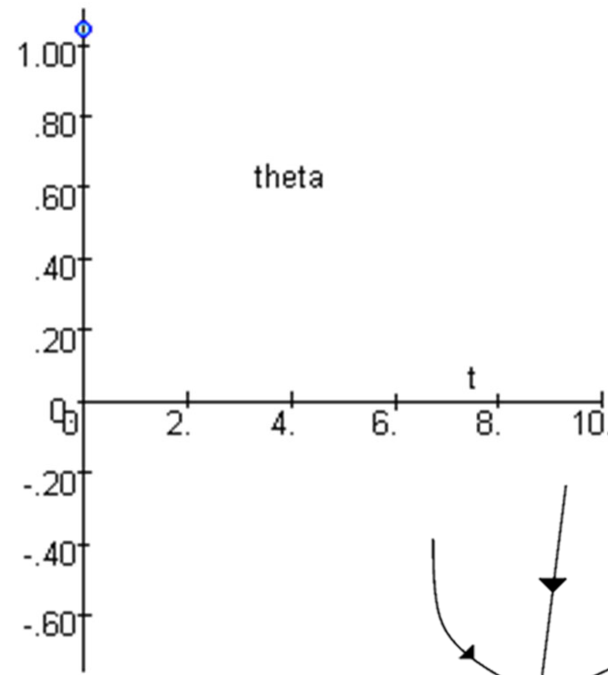
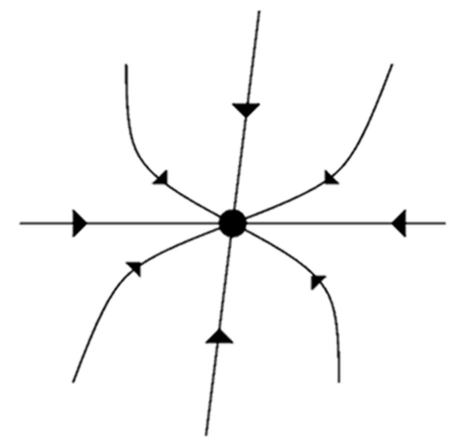
from simple…

- Attractors
  - Phase space volume to where dynamical system converges asymptotically over time
- Fixed point
  - Steady-state
  - Saddle
    - Stable in a dimension and unstable on another
    - When basins of attraction meet



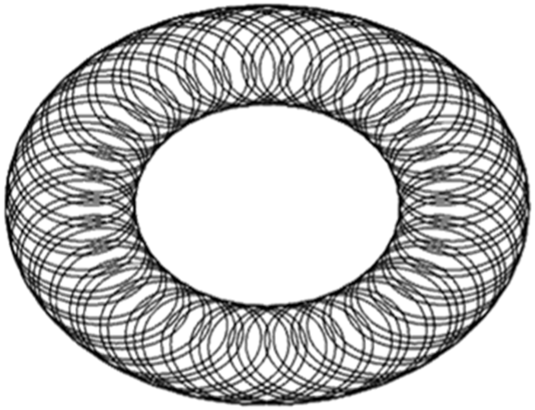


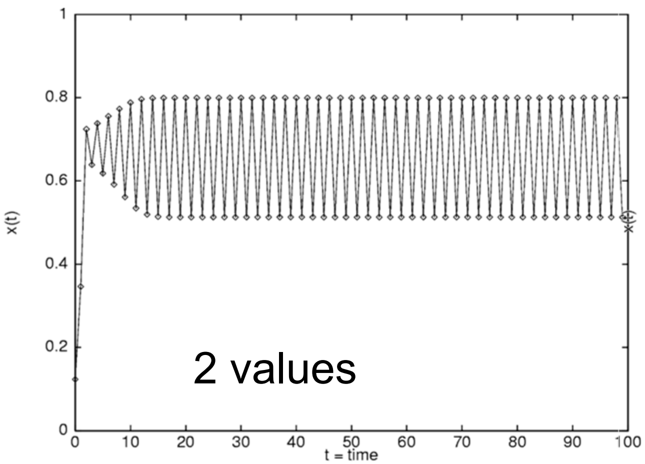


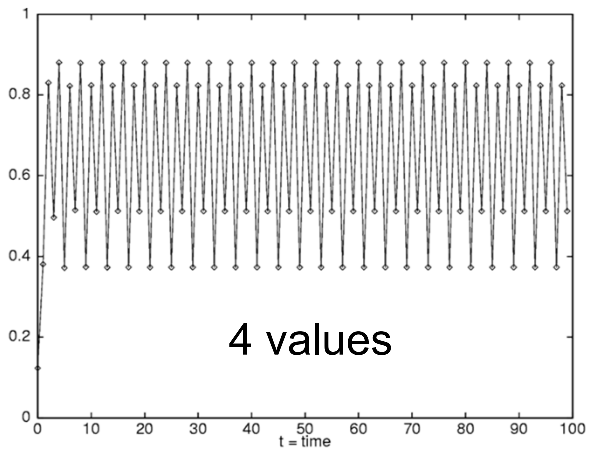


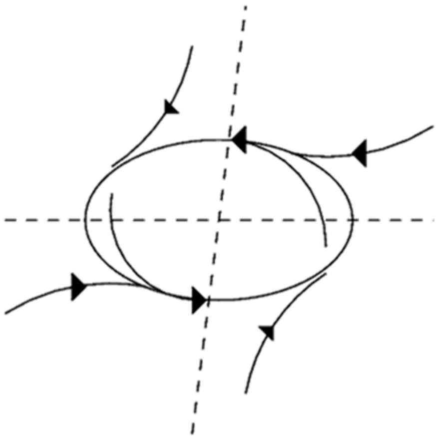
…to more complex

- Limit cycle
  - Periodic motion
  - Repetitive oscillation among a number of states
    - Loop
- Quasiperiodic attractor
  - Several independent cyclic motions
  - Toroidal attractors
  - Never quite repeat themselves









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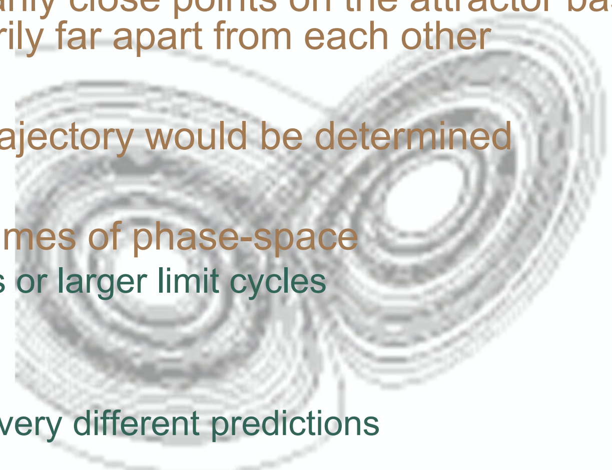
strange attractors

- Strange or chaotic attractors
  - Sensitivity to initial conditions
    - If system is released from two distinct, arbitrarily close points on the attractor basin, after sufficient time their trajectories will be arbitrarily far apart from each other
  - Deterministic Chaos
    - If we could know the exact initial condition, trajectory would be determined
  - Low-dimensional chaos
    - Strange attractors are restricted to small volumes of phase-space
      - Can be more ordered than Hamiltonian chaos or larger limit cycles
  - Weak Causality
    - 3-body problem
      - Any slight measurement difference results in very different predictions
    - Butterfly effect
      - Lorenz attractor



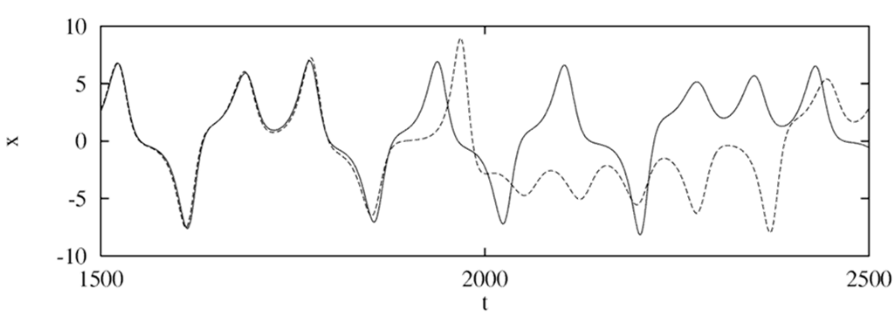
Edward Lorenz

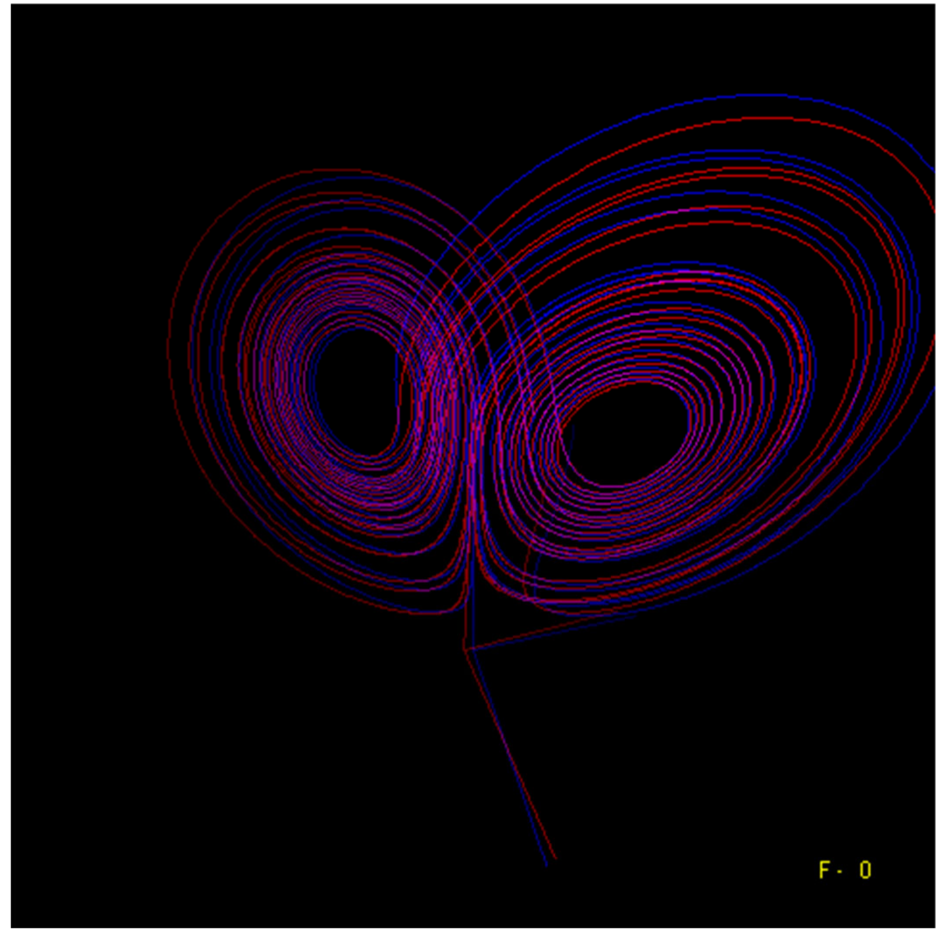
- Discovered sensitivity to initial conditions in a simple 3-variable dynamical system
  - A simplified model of weather
    - Convection flows in the atmosphere

$$\frac{dx}{dt} = \sigma(y-x)$$

$$\frac{dy}{dt} = rx-y-xz$$

$$\frac{dz}{dt} = xy-bz$$





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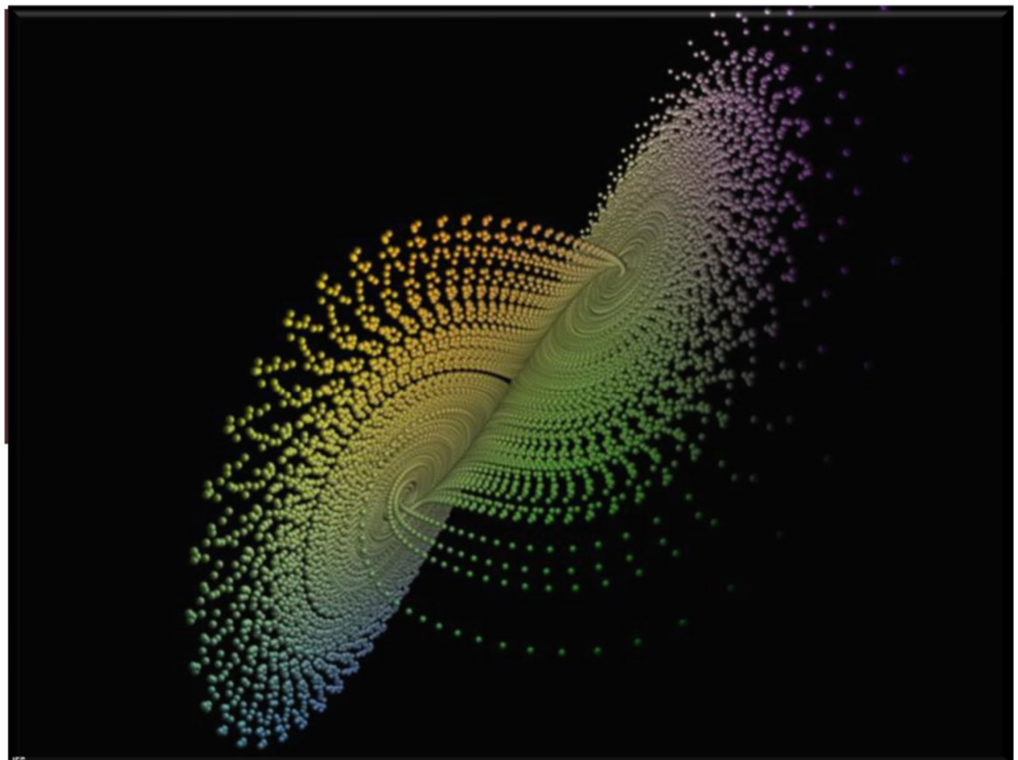
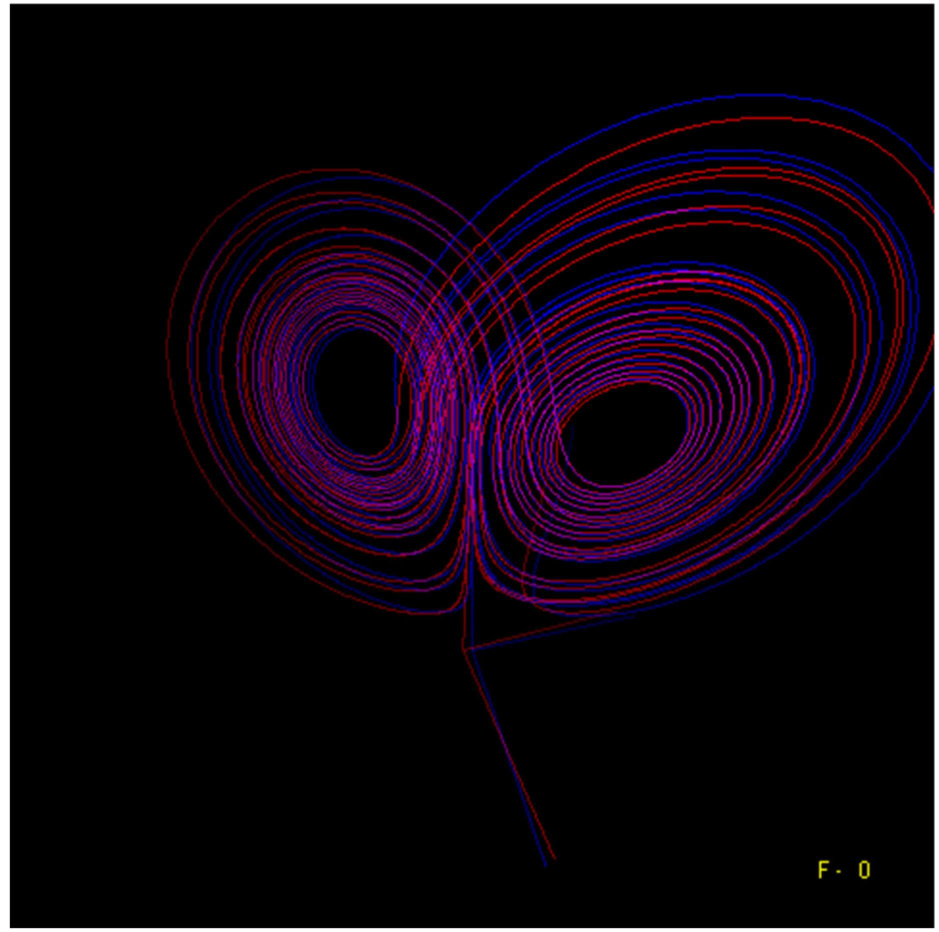




The logistic map

quadratic equation

- Demographic model
  - introduced by Pierre François Verhulst in 1838
- Continuous state-determined system
  - Memory of the previous state only
- Observations
  - X=0: population extinct
  - X=1: Overpopulation, leads to extinction

Population size

positive feedback

negative feedback

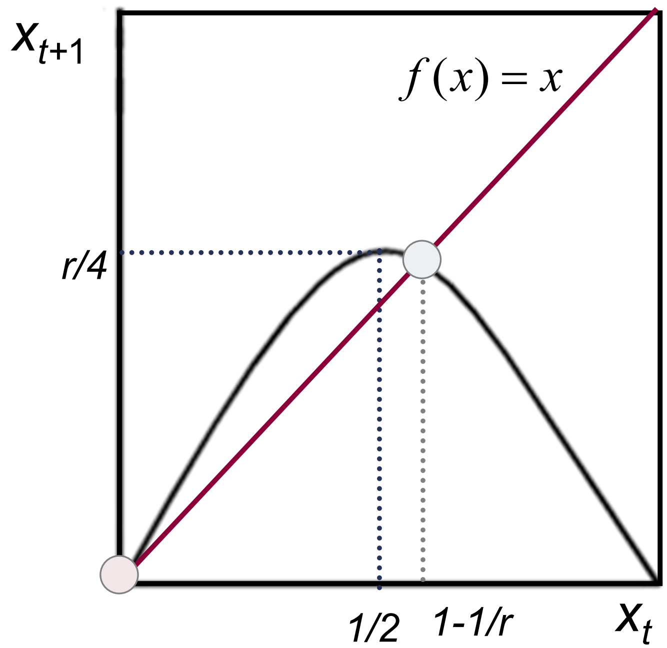
$$x_{t+1} = r x_t (1 - x_t)$$

Reproduction rate

$$x \in [0,1]$$

$$r \in [0,4]$$

plot



$$f(x) = rx(1-x)$$

$$x_{t+1} = rx_t(1-x_t)$$

$$x \in [0,1]$$

$$r \in [0,4]$$

$$f'(x) = r(1-2x), \begin{cases} |f'(x)| < 1 \Rightarrow \text{stable} \\ |f'(x)| > 1 \Rightarrow \text{unstable} \end{cases}$$

$$x = 0 \vee x = 1 - \frac{1}{r}$$

Fixed-point attractors

$$f(x) = x \Leftrightarrow rx(1-x) = x \Leftrightarrow x\big(r(x-1)+1\big) = 0$$

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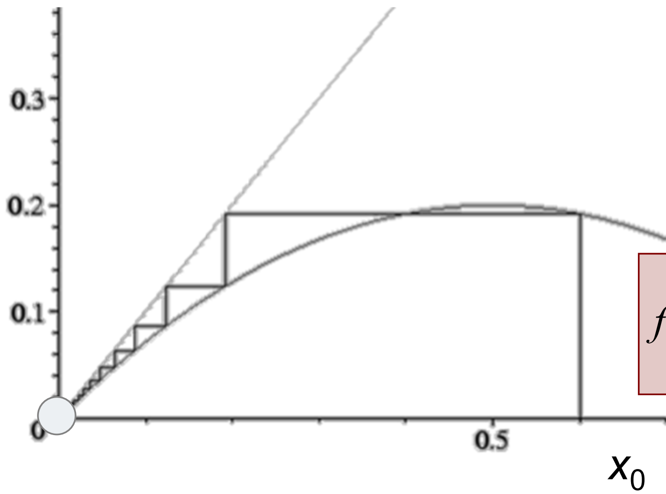
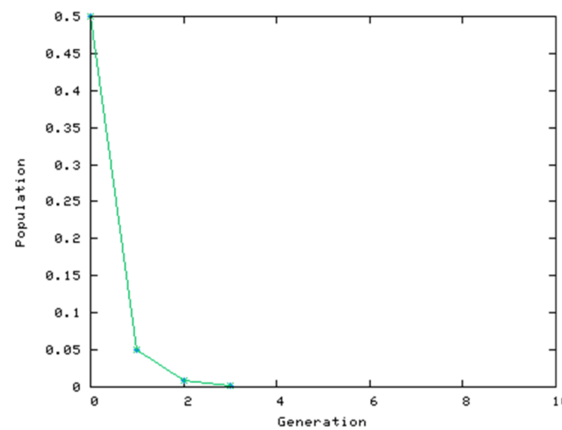
r ≤ 1 (population goes extinct)

$$x = 0 \vee \cancel{x = 1 - \frac{1}{r}}$$





$$f'(x) = r(1-2x), \begin{cases} |f'(x)| < 1 \Rightarrow \text{stable} \\ |f'(x)| > 1 \Rightarrow \text{unstable} \end{cases}$$

Stable attractor = population extinction

$$x = 0 \Rightarrow |f'(x)| = |r(1-2x)| = r, \begin{cases} r < 1 \Rightarrow \text{stable} \\ r > 1 \Rightarrow \text{unstable} \end{cases}$$

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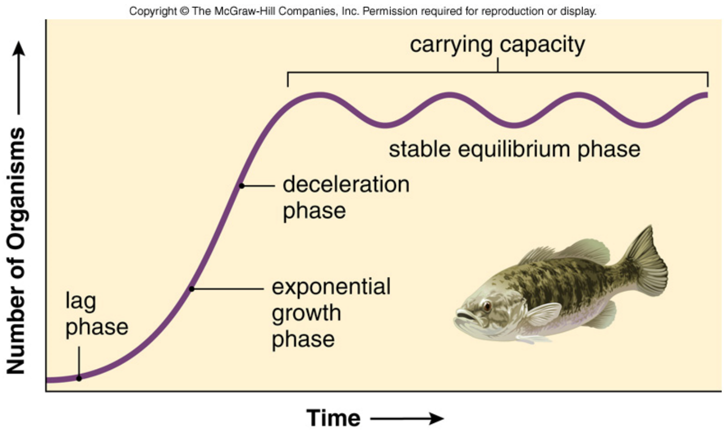
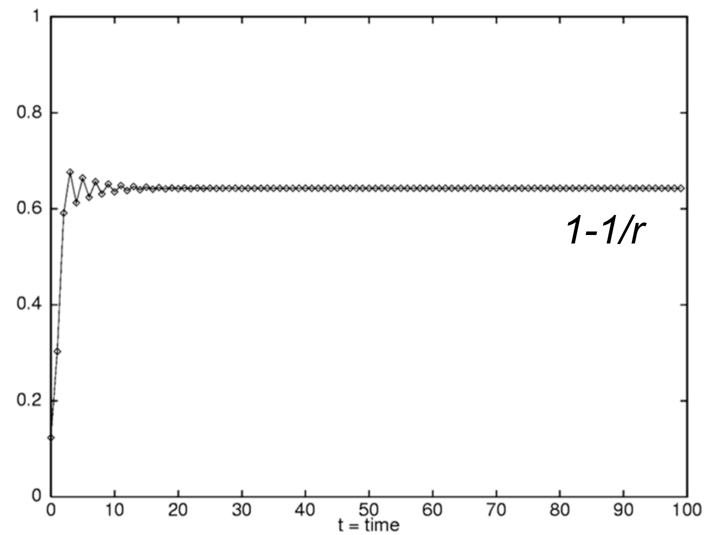
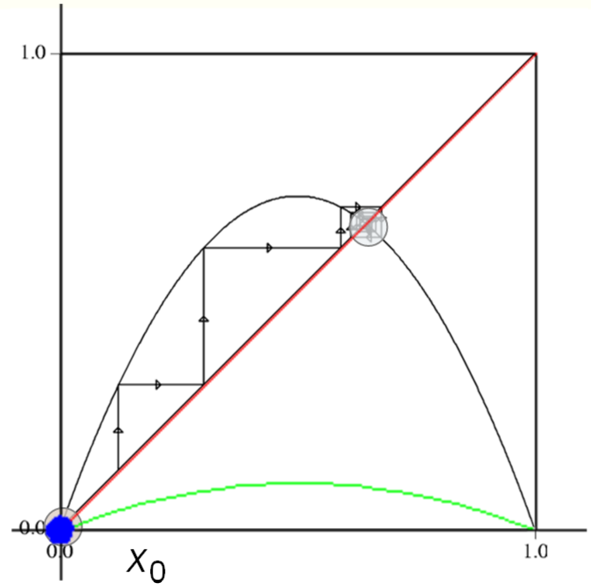
$1 \leq r \leq 3$







$$x = 0 \vee x = 1 - \frac{1}{r}$$

$$f'(x) = r(1-2x), \begin{cases} |f'(x)| < 1 \Rightarrow \text{stable} \\ |f'(x)| > 1 \Rightarrow \text{unstable} \end{cases}$$

Stable attractor = 1-1/*r*

$$x = 0 \Rightarrow |f'(x)| = r$$

$$x = 1 - \frac{1}{r} \Rightarrow |f'(x)| = |2 - r|$$

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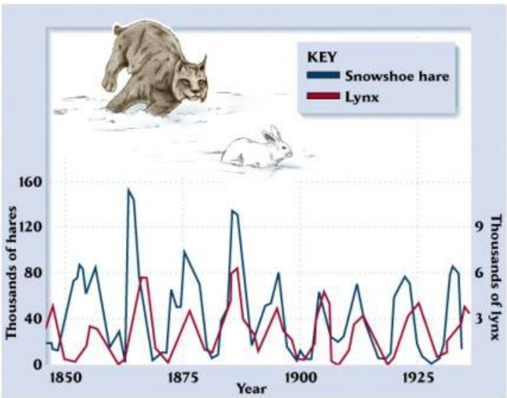
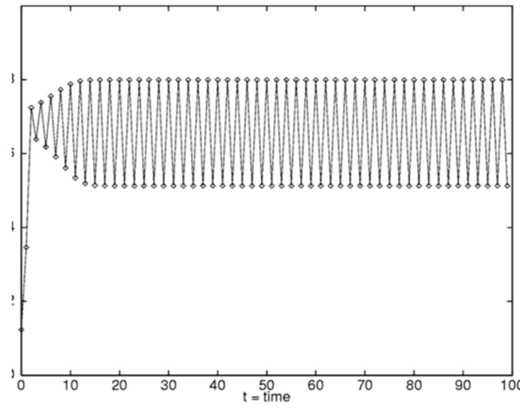
$3 \leq r \leq 4$ $(r \leq 3.44)$



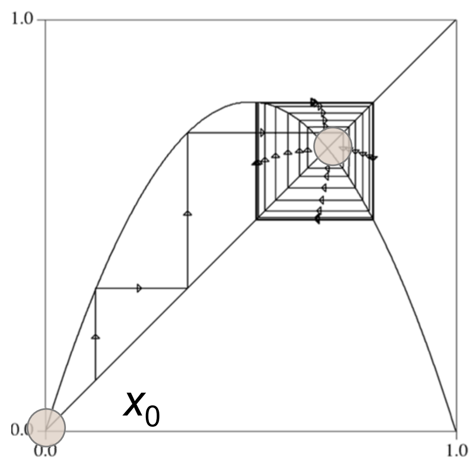


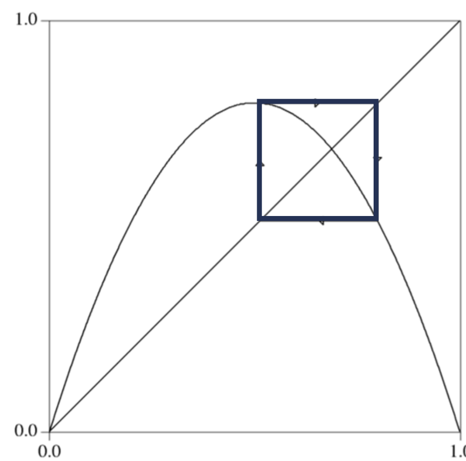
$$x = 0 \vee x = 1 - \frac{1}{r}$$

$$f'(x) = r(1-2x), \begin{cases} |f'(x)| < 1 \Rightarrow \text{stable} \\ |f'(x)| > 1 \Rightarrow \text{unstable} \end{cases}$$

$$x = 1 - \frac{1}{r} \Rightarrow |f'(x)| = |2 - r|$$

$$x = 0 \Rightarrow |f'(x)| = r$$





Limit cycle

$$f(f(x)) = x$$

Stable attractor = 2-point limit-cycle (oscillation)

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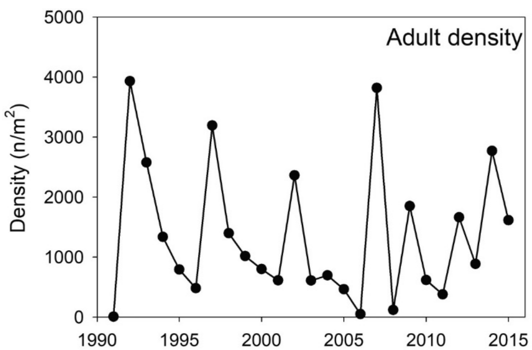
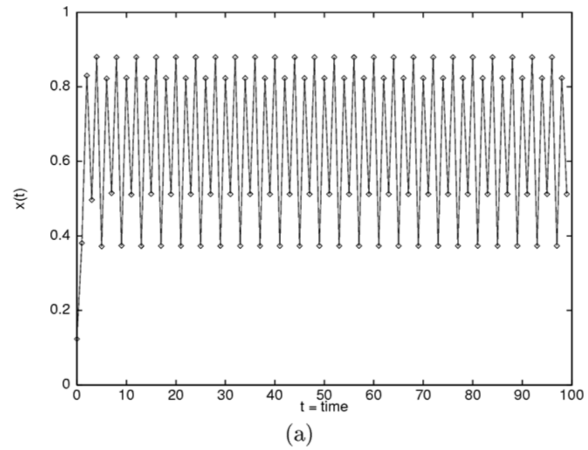
$3 \leq r \leq 4$ ($3.44 \leq r \leq 3.54$)





$$x = 0 \vee x = 1 - \frac{1}{r}$$

$$f(f(f(f(x)))) = x$$

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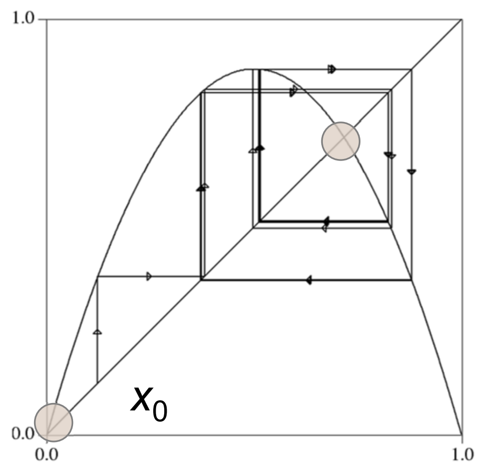

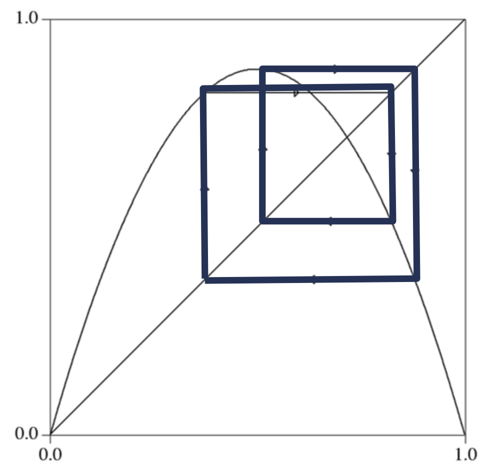

Limit cycle

Stable attractor = 4-point limit-cycle

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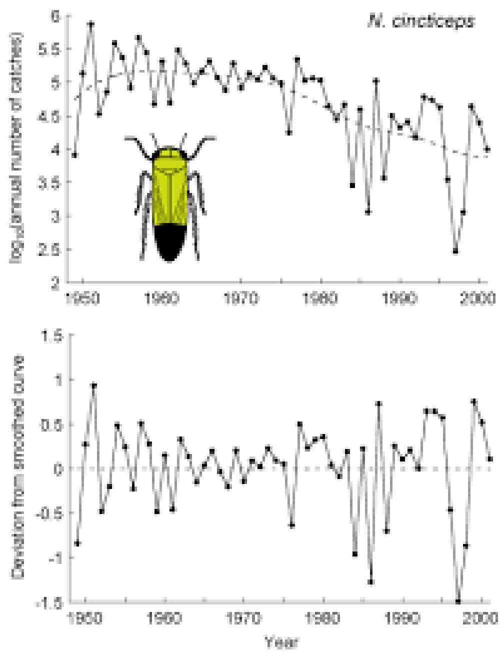
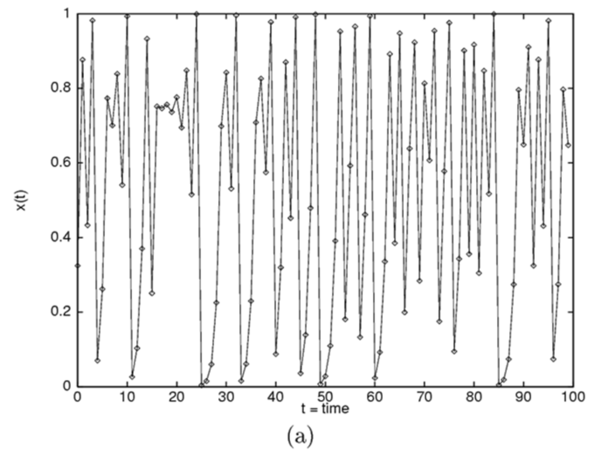
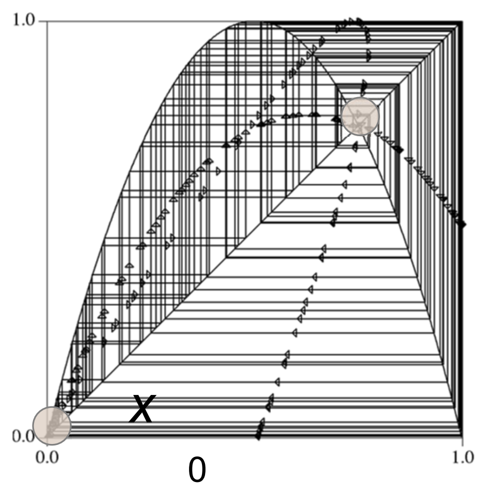
logistic map

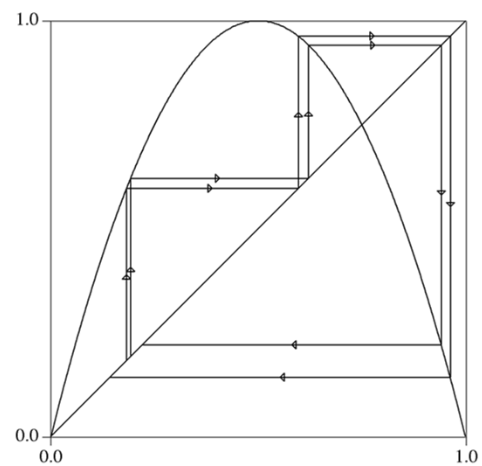
r = 4





$$x = 0 \vee x = 1 - \frac{1}{r}$$





Chaotic

Deterministic

Sensitive

ergodic

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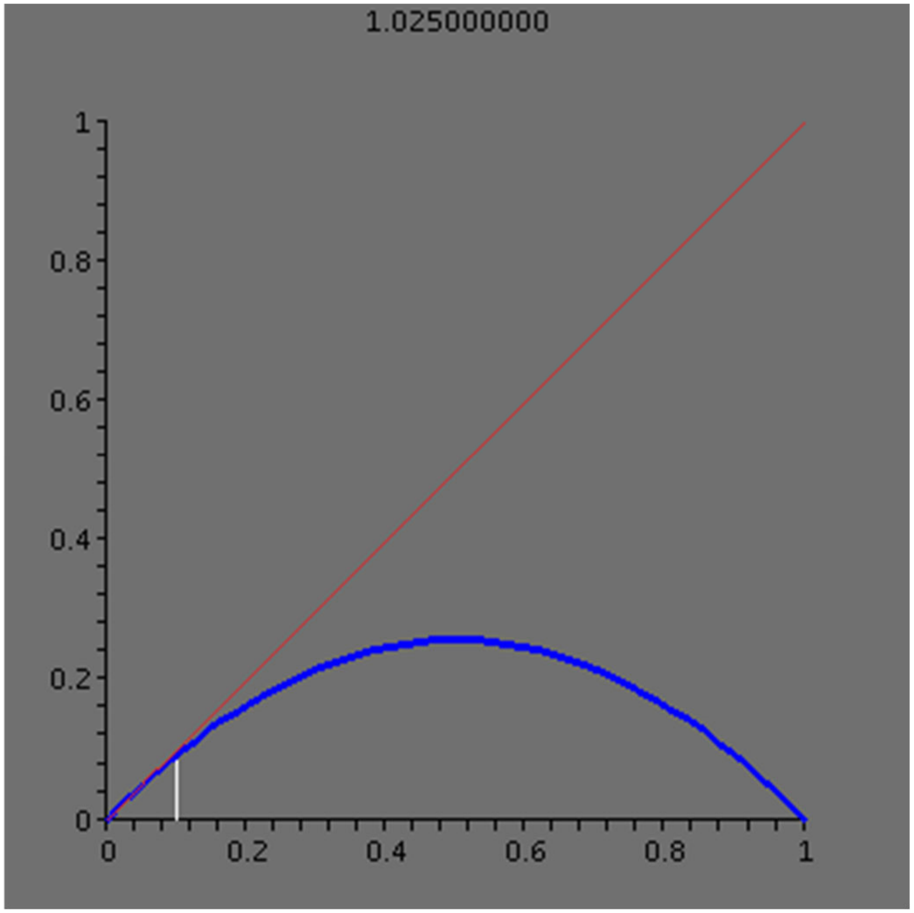
movie



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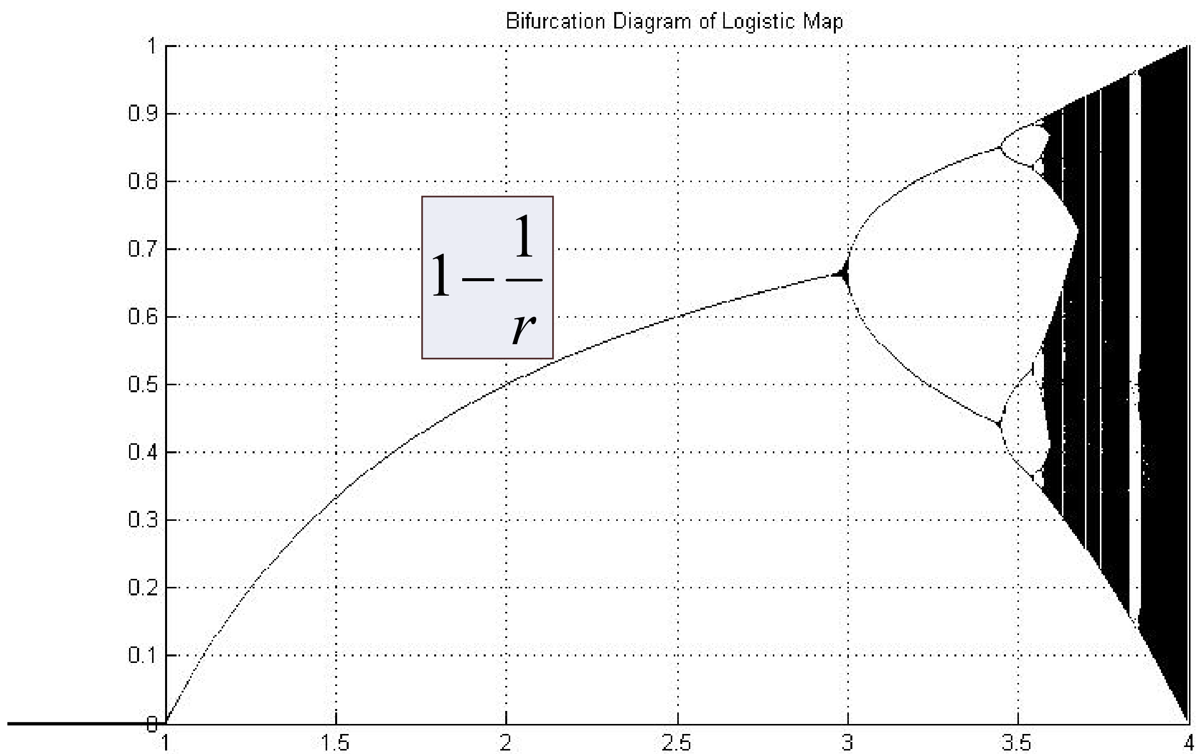
bifurcation map



Period doubling

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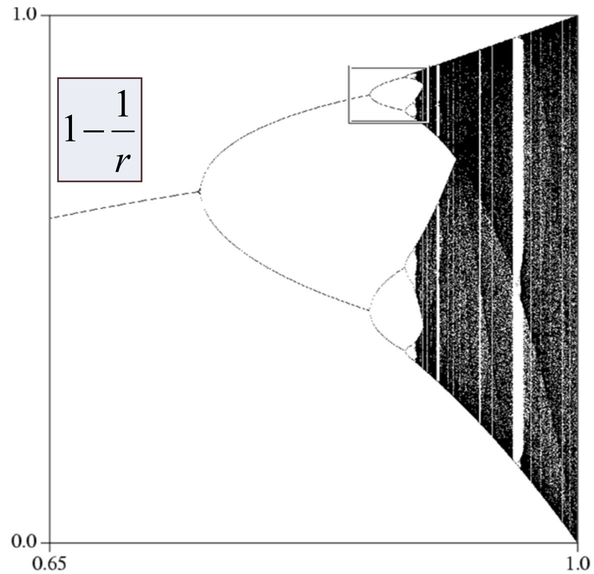
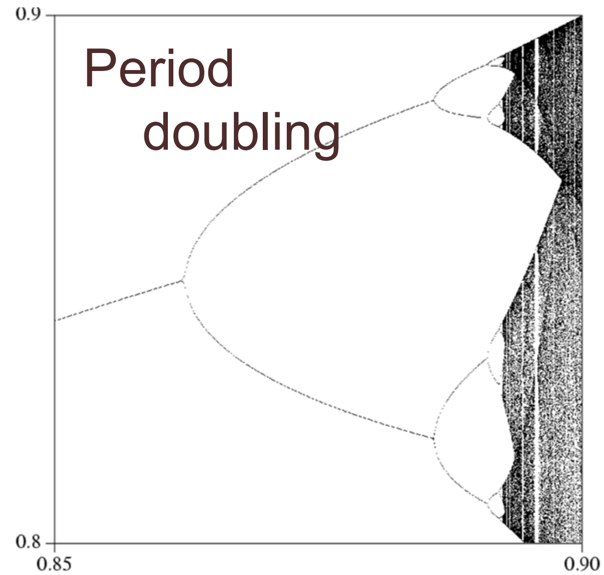
bifurcation map





Chaotic

Deterministic (not random)

Sensitive

ergodic

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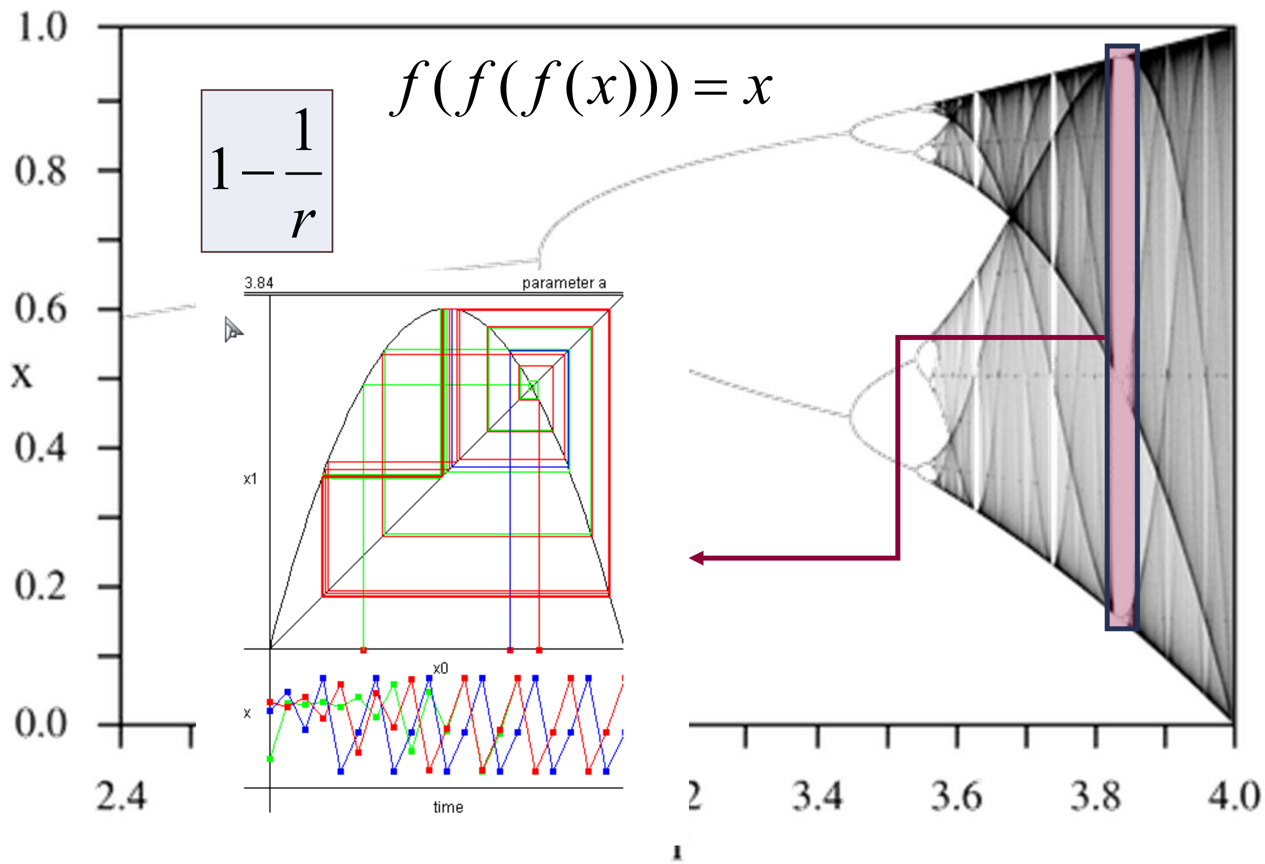
bifurcation map: cycle of 3



Period doubling

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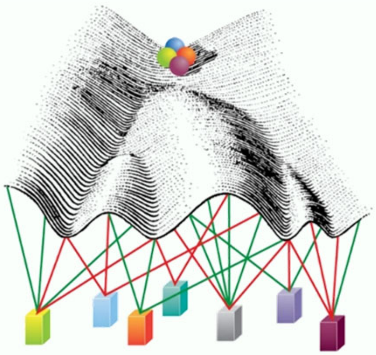
readings

- Class Book
  - Floreano, D. and C. Mattiussi [2008]. *Bio-Inspired Artificial Intelligence: Theories, Methods, and Technologies*. MIT Press.
    - **Chapter 2**.
- Lecture notes
  - Chapter 1: What is Life?
  - Chapter 2: The logical Mechanisms of Life
  - Chapter 3: Formalizing and Modeling the World
    - posted online @ http://informatics.indiana.edu/rocha/i-bic



- Papers and other materials
  - Discussions
    - Scheffer, Marten, et al. "Early-warning signals for critical transitions." *Nature* **461**.7260 [2009]: 53.
      - Optional: Leemput, et al. [2014] "Critical Slowing down as Early Warning for the Onset and Termination of Depression." PNAS. **111** (1): 87–92.
    - Manicka, S., M. Marques-Pita, and L.M. Rocha [2022]. Effective connectivity determines the critical dynamics of biochemical networks. *Journal of the Royal Society Interface*. **19**(186):20210659.
    - Lindgren, K. [1991]."Evolutionary Phenomena in Simple Dynamics." In: *Artificial Life II*. Langton et al (Eds). Addison-wesley, pp. 295-312.
      - Optional: Hinton, G.E. and S.J. Nowlan [1987]."How learning can guide evolution." *Complex Systems*. **1,** pp.495-502.
    - Papadopoulou et al [2022]. "Self-Organization of Collective Escape in Pigeon Flocks." *PLOS Computational Biology* **18** (1): e1009772.
  - Optional
    - Nunes de Castro, Leandro [2006]. *Fundamentals of Natural Computing: Basic Concepts, Algorithms, and Applications*. Chapman & Hall.
      - Chapter 2, Section 7.3, Sections 8.1, 8.2, 8.3.10
    - Flake's [1998], *The Computational Beauty of Life*. MIT Press.
      - Chapters 10, 11, 14 – Dynamics, Attractors and chaos